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Convexity Properties of
Hamiltonian Group Actions

Victor Guillemin
Reyer Sjamaar



American Mathematical Society

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Convexity Properties of Hamiltonian Group Actions

Victor Guillemin
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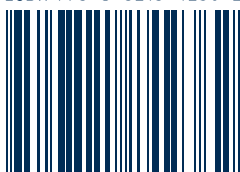
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This is a monograph on convexity properties of moment mappings in symplectic geometry. The fundamental result in this subject is the Kirwan convexity theorem, which describes the image of a moment map in terms of linear inequalities. This theorem bears a close relationship to perplexing old puzzles from linear algebra, such as the Horn problem on sums of Hermitian matrices, on which considerable progress has been made in recent years following a breakthrough by Klyachko. The book presents a simple local model for the moment polytope, valid in the “generic” case, and an elementary Morse-theoretic argument deriving the Klyachko inequalities and some of their generalizations. It reviews various infinite-dimensional manifestations of moment convexity, such as the Kostant type theorems for orbits of a loop group (due to Atiyah and Pressley) or a symplectomorphism group (due to Bloch, Flaschka and Ratiu). Finally, it gives an account of a new convexity theorem for moment map images of orbits of a Borel subgroup of a complex reductive group acting on a Kähler manifold, based on potential-theoretic methods in several complex variables.

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