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American Mathematical Society

https://doi.org/10.1090/crmm/026

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American Mathematical Society
Providence, Rhode Island USA

The production of this volume was supported in part by the Fonds Québécois de la Recherche sur la Nature et les Technologies (FQRNT) and the Natural Sciences and Engineering Research Council of Canada (NSERC).

2000 *Mathematics Subject Classification*. Primary 53D20; Secondary 14L24, 53–02, 53C55.

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Library of Congress Cataloging-in-Publication Data

Guillemin, V., 1937–
Convexity properties of Hamiltonian group actions / Victor Guillemin, Reyer Sjamaar.
p. cm. — (CRM monograph series, ISSN 1065-8599; v. 26)
Includes bibliographical references.
ISBN 0-8218-3918-7 (hard cover; alk. paper); ISBN 0-8218-4236-6 (soft cover; alk. paper)
1. Hamiltonian systems. 2. Convex domains. 3. Matrices. 4. Convex functions. 5. Convexity spaces. I. Sjamaar, Reyer, 1961– II. Title. III. Series.

QA614.83.G85 2005
515′.39—dc22

2005044490

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
This volume was submitted to the American Mathematical Society
in camera-ready form by the Centre de Recherches Mathématiques.
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10 9 8 7 6 5 4 3 2 17 16 15 14 13 12

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Bibliography

[1] S. Agnihotri and C. Woodward, *Eigenvalues of products of unitary matrices and quantum Schubert calculus*, Math. Res. Lett. **5** (1998), no. 6, 817–836.

[2] A. Alekseev, *On Poisson actions of compact Lie groups on symplectic manifolds*, J. Differential Geom. **45** (1997), no. 2, 241–256.

[3] A. Alekseev, A. Malkin, and E. Meinrenken, *Lie group valued moment maps*, J. Differential Geom. **48** (1998), no. 3, 445–495.

[4] M. Atiyah, *Convexity and commuting Hamiltonians*, Bull. London Math. Soc. **14** (1982), no. 1, 1–15.

[5] ———, *Circular symmetry and stationary-phase approximation*, Colloque en l'honneur de Laurent Schwartz, vol. 1 (Palaiseau, 1983), Astérisque **131** (1985), 43–59.

[6] M. Atiyah and A. Pressley, *Convexity and loop groups*, Arithmetic and geometry, vol. II (M. Artin and J. Tate, eds.), Progr. Math. **36**, Birkhäuser, Boston, 1983, pp. 33–63.

[7] M. Audin, *The topology of torus actions on symplectic manifolds*, Progress in Mathematics, vol. 93, Birkhäuser-Verlag, Basel, 1991, Translated from the French by the author.

[8] P. Belkale, *Local systems on $\mathbb{P}^1 - S$ for S a finite set*, Compositio Math. **129** (2001), no. 1, 67–86.

[9] A. Berenstein and R. Sjamaar, *Coadjoint orbits, moment polytopes, and the Hilbert–Mumford criterion*, J. Amer. Math. Soc. **13** (2000), no. 2, 433–466.

[10] A. Białynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. of Math. (2) **98** (1973), 480–497.

[11] ———, *Some properties of the decompositions of algebraic varieties determined by actions of a torus*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **24** (1976), no. 9, 667–674.

[12] J.-M. Bismut and W. Zhang, *An extension of a theorem by Cheeger and Müller*, Astérisque **205** (1992).

[13] A. Bloch, H. Flaschka, and T. Ratiu, *A Schur–Horn–Kostant convexity theorem for the diffeomorphism group of the annulus*, Invent. Math. **113** (1993), no. 3, 511–529.

[14] N. Bourbaki, *Groupes et algèbres de Lie*, Éléments de mathématique, Masson, Paris, 1982.

[15] G. Bredon, *Introduction to compact transformation groups*, Pure Appl. Math., vol. 46, Academic Press, New York, 1972.

[16] M. Brion, *Sur l'image de l'application moment*, Séminaire d'Algèbre Paul Dubreil et Marie-Paule Malliavin (Paris, 1986) (M.-P. Malliavin, ed.), Lecture Notes in Math., vol. 1296, Springer-Verlag, Berlin, 1987, pp. 177–192.

[17] D. Burns and V. Guillemin, *Potential functions and actions of tori on Kähler manifolds*, Comm. Anal. Geom. **12** (2004), no. 1–2, 281–303.

[18] A. Cannas da Silva, *Lectures on symplectic geometry*, Lecture Notes in Math., vol. 1764, Springer-Verlag, Berlin, 2001.

[19] J. Carrell and A. Sommese, $\mathbf{C}^*$-*actions*, Math. Scand. **43** (1978/79), no. 1, 49–59.

[20] ———, *Correction*, ibid. **53**, no. 1 (1983), 32.

[21] ———, *Some topological aspects of* $\mathbf{C}^*$ *actions on compact Kaehler manifolds*, Comment. Math. Helv. **54** (1979), no. 4, 567–582.

[22] T. Delzant, *Hamiltoniens périodiques et images convexes de l'application moment*, Bull. Soc. Math. France **116** (1988), no. 3, 315–339.

[23] ———, *Classification des actions hamiltoniennes complètement intégrables de rang deux*, Ann. Global Anal. Geom. **8** (1990), no. 1, 87–112.

[24] M. Demazure, *Sous-groupes algébriques de rang maximum du groupe de Cremona*, Ann. Sci. École Norm. Sup. (4) **3** (1970), 507–588.

[25] J. J. Duistermaat, *Convexity and tightness for restrictions of Hamiltonian functions to fixed point sets of an antisymplectic involution*, Trans. Amer. Math. Soc. **275** (1983), no. 1, 417–429.

[26] H. Flaschka and T. Ratiu, *A convexity theorem for Poisson actions of compact Lie groups*, Ann. sci. École norm. sup. (4) **29** (1996), no. 6, 787–809.

[27] W. Fulton, *Eigenvalues, invariant factors, highest weights, and Schubert calculus*, Bull. Amer. Math. Soc. (N.S.) **37** (2000), no. 3, 209–249.

[28] V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), no. 3, 491–513.

[29] ———, *The Gelfand–Cetlin system and quantization of the complex flag manifolds*, J. Funct. Anal. **52** (1983), no. 1, 106–128.

[30] ———, *Convexity properties of the moment mapping.* II, Invent. Math. **77** (1984), no. 3, 533–546.

[31] ———, *Multiplicity-free spaces*, J. Differential Geom. **19** (1984), no. 1, 31–56.

[32] ———, *A normal form for the moment map*, Differential Geometric Methods in Mathematical Physics (Jerusalem, 1982) (S. Sternberg, ed.), Mathematical Physics Studies, vol. 6, D. Reidel Publishing Company, Dordrecht, 1984, pp. 161–175.

[33] G. Hardy, J. Littlewood, and G. Pólya, *Inequalities*, 2nd ed., Cambridge University Press, 1952.

[34] F. R. Harvey and H. B. Lawson, Jr., *Morse theory and Stokes' theorem*, Papers dedicated to Atiyah, Bott, Hirzebruch, and Singer (S.-T. Yau, ed.), Surv. Differ. Geom., vol. VII, Int. Press, Somerville, MA, 2000, pp. 259–311.

[35] ———, *Finite volume flows and Morse theory*, Ann. of Math. (2) **153** (2001), no. 1, 1–25.

[36] A. Horn, *Doubly stochastic matrices and the diagonal of a rotation matrix*, Amer. J. Math. **76** (1954), 620–630.

[37] P. Iglésias, *Les* SO(3)*-variétés symplectiques et leur classification en dimension* 4, Bull. Soc. Math. France **119** (1991), no. 3, 371–396.

[38] V. Kač and D. Peterson, *Unitary structure in representations of infinite-dimensional groups and a convexity theorem*, Invent. Math. **76** (1984), no. 1, 1–14.

[39] Y. Karshon, *Periodic Hamiltonian flows on four-dimensional manifolds*, Mem. Amer. Math. Soc., vol. 141, Amer. Math. Soc., Providence, R.I., 1999.

[40] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal embeddings.* I, Lecture Notes in Math., vol. 339, Springer-Verlag, Berlin, 1973.

[41] F. Kirwan, *Convexity properties of the moment mapping.* III, Invent. Math. **77** (1984), no. 3, 547–552.

[42] A. Klyachko, *Stable bundles, representation theory and Hermitian operators*, Selecta Math. (N.S.) **4** (1998), no. 3, 419–445.

[43] A. Knutson and T. Tao, *The honeycomb model of* $\mathrm{GL}_n(\mathbf{C})$ *tensor products.* I. *Proof of the saturation conjecture*, J. Amer. Math. Soc. **12** (1999), no. 4, 1055–1090.

[44] A. Knutson, T. Tao, and C. Woodward, *The honeycomb model of* $mathrmGL_n(\mathbf{C})$ *tensor products.* II. *Puzzles determine facets of the Littlewood–Richardson cone*, J. Amer. Math. Soc. **17** (2004), no. 1, 19–48.

[45] B. Kostant, *Quantization and unitary representations. I. Prequantization*, Lectures in Modern Analysis and Applications, III (Washington, D.C.) (C. T. Taam, ed.), Lecture Notes in Math., vol. 170, Springer-Verlag, Berlin, 1970, pp. 87–208.

[46] ______, *On convexity, the Weyl group and the Iwasawa decomposition*, Ann. sci. École norm. sup. (4) **6** (1973), 413–455.

[47] M. Lassalle, *Deux généralisations du "théorème des trois cercles" de Hadamard*, Math. Ann. **249** (1980), no. 1, 17–26.

[48] E. Lerman, *Symplectic cuts*, Math. Res. Lett. **2** (1995), no. 3, 247–258.

[49] E. Lerman, E. Meinrenken, S. Tolman, and C. Woodward, *Nonabelian convexity by symplectic cuts*, Topology **37** (1998), no. 2, 245–259.

[50] J.-H. Lu, *Momentum mappings and reduction of Poisson actions*, Symplectic geometry, groupoids, and integrable systems (Berkeley, CA, 1989) (P. Dazord and A. Weinstein, eds.), Math. Sci. Res. Inst. Publ., vol. 20, Springer, New York, 1991, pp. 209–226.

[51] J.-H. Lu and A. Weinstein, *Poisson Lie groups, dressing transformations, and Bruhat decompositions*, J. Differential Geom. **31** (1990), no. 2, 501–526.

[52] C.-M. Marle, *Modèle d'action hamiltonienne d'un groupe de Lie sur une variété symplectique*, Rend. Sem. Mat. Univ. Politec. Torino **43** (1985), no. 2, 227–251.

[53] E. Meinrenken and C. Woodward, *Hamiltonian loop group actions and Verlinde factorization*, J. Differential Geom. **50** (1998), no. 3, 417–469.

[54] K.-H. Neeb, *Convexity properties of the coadjoint action of non-compact Lie groups*, Math. Ann. **309** (1997), no. 4, 625–661.

[55] L. O'Shea and R. Sjamaar, *Moment maps and Riemannian symmetric pairs*, Math. Ann. **317** (2000), no. 3, 415–457.

[56] S. Paneitz, *Invariant convex cones and causality in semisimple Lie algebras and groups*, J. Funct. Anal. **43** (1981), no. 3, 313–359.

[57] E. Prato, *Convexity properties of the moment map for certain non-compact manifolds*, Comm. Anal. Geom. **2** (1994), no. 2, 267–278.

[58] A. Pressley and G. Segal, *Loop groups*, Oxford Mathematical Monographs, Oxford University Press, New York, 1986.

[59] G. de Rham, *Variétés différentiables. Formes, courants, formes harmoniques*, 3rd revised and augmented ed., Publications de l'Institut de Mathématique de l'Université de Nancago III, Actualités Scientifiques et Industrielles, Hermann, Paris, 1973.

[60] R. Sjamaar, *Convexity properties of the moment mapping re-examined*, Adv. Math. **138** (1998), no. 1, 46–91.

[61] C.-L. Terng, *Convexity theorem for infinite-dimensional isoparametric submanifolds*, Invent. Math. **112** (1993), no. 1, 9–22.

[62] A. Thimm, *Integrable geodesic flows on homogeneous spaces*, Ergodic Theory Dynamical Systems **1** (1981), no. 4, 495–517 (1982).

[63] S. Tolman, *Examples of non-Kähler Hamiltonian torus actions*, Invent. Math. **131** (1998), no. 2, 299–310.

[64] A. Weinstein, *Neighborhood classification of isotropic embeddings*, J. Differential Geom. **16** (1981), no. 1, 125–128.

[65] ______, *Poisson geometry of discrete series orbits, and momentum convexity for noncompact group actions*, Lett. Math. Phys. **56** (2001), no. 1, 17–30, EuroConférence Moshé Flato, Part I (Dijon, 2000).

[66] H. Wielandt, *An extremum property of sums of eigenvalues*, Proc. Amer. Math. Soc. **6** (1955), 106–110.

[67] C. Woodward, *The classification of transversal multiplicity-free group actions*, Ann. Global Anal. Geom. **14** (1996), no. 1, 3–42.

[68] ______, *Multiplicity-free Hamiltonian actions need not be Kähler*, Invent. Math. **131** (1998), no. 2, 311–319.

[69] ———, *Spherical varieties and existence of invariant Kähler structures*, Duke Math. J. **93** (1998), no. 2, 345–377.

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ISBN 978-0-8218-4236-2

9 780821 842362

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