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Skew-Orthogonal Polynomials and Random Matrix Theory

Saugata Ghosh

American Mathematical Society

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Saugata Ghosh

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APPENDIX A

Proofs of (5.7), (5.12), and (5.19)

Proof of (5.7). The asymptotic form of the associated Laguerre polynomial is given by (5.6). For convenience, we call the amplitude part $A(\theta, m)$ and the argument part $f_m(\theta) + \alpha$. Thus (5.6) can be rewritten as

$$\phi_m^{(2)}(x) \equiv (h_m^{(a)})^{-1/2}\big(w_a(x)\big)^{1/2} L_m^{(a)}(x) = A(\theta, m)\sin[f_m(\theta_m) + \alpha], \tag{A.1}$$

where

$$\begin{aligned} A(\theta, m) &= \frac{(-1)^m}{\sqrt{2\pi \sin\theta_m \cos\theta_m}}, \\ f_m(\theta_m) &= \left(m + \frac{a+1}{2}\right)(\sin 2\theta_m - 2\theta_m), \\ \alpha &= \frac{3\pi}{4}, \end{aligned}$$

where $x = (4m + 2a + 2)\cos^2\theta$. Here, x does not depend on m but θ does. Hence differentiating with respect to θ and putting $\Delta m = \pm 1$, we have

$$\Delta\theta_m = \pm\frac{1}{2}\left[\left(m + \frac{a+1}{2}\right)\tan\theta_m\right]^{-1} \simeq \pm\frac{1}{2m\tan\theta_m} \tag{A.2}$$

We also get

$$\left(\frac{\partial f_m(\theta_m)}{\partial m}\right) = (\sin 2\theta_m - 2\theta_m) \tag{A.3}$$

and

$$\left(\frac{\partial f_m(\theta_m)}{\partial \theta_m}\right) = -4\left(m + \frac{a+1}{2}\right)\sin^2\theta_m. \tag{A.4}$$

Using (A.1)–(A.4), we can write

$$\begin{aligned} \phi_{m\pm1}^{(2)}(x) &= A(\theta_m, m)\sin[f_{m\pm1}(\theta_{m\pm1}) + \alpha] \\ &= A(\theta_m, m)\sin\left[f_m(\theta_m) \pm \frac{\partial f_m(\theta_m)}{\partial m}\Delta m \pm \frac{\partial f_m(\theta_m)}{\partial \theta_m}\Delta\theta_m + \alpha\right] \\ &= A(\theta_m, m)\sin\Bigg[\left(m + \frac{a+1}{2}\right)(\sin 2\theta_m - 2\theta_m) \\ &\qquad \pm (\sin 2\theta_m - 2\theta_m) \mp \sin 2\theta_m + \frac{3\pi}{4}\Bigg]. \end{aligned} \tag{A.5}$$

Thus we have

$$\begin{aligned} &\phi_{m\pm1}^{(2)}(x) \\ &= \frac{(-1)^m}{\sqrt{2\pi\sin\theta_m\cos\theta_m}}\sin\left[\left(m + \frac{a+1}{2}\right)(\sin 2\theta_m - 2\theta_m) \mp 2\theta_m + \frac{3\pi}{4}\right]. \quad \square \end{aligned} \tag{A.6}$$

PROOF OF (5.12). The asymptotic form of the Hermite polynomial is given by (5.11). For convenience, we call the amplitude part $A(\theta, m)$ and the argument part $f_m(\theta) + \alpha$. Thus (5.11) can be rewritten as

$$\phi_m^{(2)}(x) \equiv (h_m^{(a)})^{-1/2}\big(w_G(x)\big)^{1/2} H_m(x) = A(\theta, m)\sin[f_m(\theta_m) + \alpha], \tag{A.7}$$

where

$$\begin{aligned} A(\theta, m) &= \frac{1}{\sqrt{\pi \sin\theta}}\left(\frac{2}{m}\right)^{1/4}, \\ f_m(\theta_m) &= \left(\frac{m}{2} + \frac{1}{4}\right)(\sin 2\theta_m - 2\theta_m), \\ \alpha &= \frac{3\pi}{4} \end{aligned}$$

where $x = \sqrt{(2m+1)}\cos\theta$. Here, x does not depend on m but θ does. Hence differentiating with respect to θ and putting $\Delta m = \pm 1$, we have

$$\Delta\theta_m = \pm\frac{1}{(2m+1)\tan\theta_m} \simeq \pm\frac{1}{2m\tan\theta_m}. \tag{A.8}$$

We also get

$$\left(\frac{\partial f_m(\theta_m)}{\partial m}\right) = \frac{1}{2}(\sin 2\theta_m - 2\theta_m) \tag{A.9}$$

and

$$\left(\frac{\partial f_m(\theta_m)}{\partial \theta_m}\right) = -4\left(\frac{m}{2} + \frac{1}{4}\right)\sin^2\theta_m. \tag{A.10}$$

Using (A.7) – (A.10), we can write

$$\begin{aligned} \phi_{m\pm1}^{(2)}(x) &= A(\theta_m, m)\sin[f_{m\pm1}(\theta_{m\pm1}) + \alpha] \\ &= A(\theta_m, m)\sin\left[f_m(\theta_m) \pm \frac{\partial f_m(\theta_m)}{\partial m}\Delta m \pm \frac{\partial f_m(\theta_m)}{\partial \theta_m}\Delta\theta_m + \alpha\right] \\ &= A(\theta_m, m)\sin\Bigg[\left(\frac{m}{2} + \frac{1}{4}\right)(\sin 2\theta_m - 2\theta_m) \\ &\qquad \pm \frac{1}{2}(\sin 2\theta_m - 2\theta_m) \mp \frac{\sin 2\theta_m}{2} + \frac{3\pi}{4}\Bigg]. \end{aligned} \tag{A.11}$$

Thus we have

$$\begin{aligned} &\phi_{m\pm1}^{(2)}(x) \\ &= \frac{1}{\sqrt{\pi\sin\theta_m}}\left(\frac{2}{m}\right)^{1/4}\sin\left[\left(\frac{j}{2} + \frac{1}{4}\right)(\sin 2\theta_m - 2\theta_m) \mp \theta_m + \frac{3\pi}{4}\right]. \quad \square \end{aligned} \tag{A.12}$$

PROOF OF (5.19). The asymptotic form of orthogonal functions corresponding to the quartic weight is given by (5.16). For convenience, we call the amplitude part $A(x, \theta_m)$ and the argument part $f_m(\theta)$.

$$\phi_m^{(2)}(x) = A(x, \theta_m)\big[\cos\big(f_m(\theta)\big) + O(N^{-1})\big], \tag{A.13}$$

where

$$f_m(\theta) = \left(\frac{m + \frac{1}{2}}{2}\right)\left(\frac{\sin(2\theta)}{2} - \theta\right) - (-1)^m\frac{\chi}{4} + \frac{\pi}{4}.$$

We know that

$$2\sqrt{\lambda' g}\cos\theta = gx^2 + t, \qquad \lambda' = \frac{m+\frac{1}{2}}{N}, \qquad \lambda \equiv \frac{m}{N}, \tag{A.14}$$

where, for a given x, θ varies with λ. Hence differentiating with respect to θ and putting $N\Delta\lambda' = \pm 1$, we have

$$\Delta\theta_m = \pm\frac{1}{2\lambda'\tan\theta_m}. \tag{A.15}$$

We also get

$$\left(\frac{\partial f_m(\theta_m)}{\partial\lambda'}\right) = \frac{N}{2}\left(\frac{\sin 2\theta_m}{2} - \theta_m\right) \tag{A.16}$$

and

$$\left(\frac{\partial f_m(\theta_m)}{\partial\theta_m}\right) = -N\lambda'\sin^2\theta_m. \tag{A.17}$$

Using (A.15)–(A.17), we can write

$$\begin{aligned}\phi^{(2)}_{m\pm1}(x) &= A(x,\theta_m)\cos[f_{m\pm1}(\theta_{m\pm1})] \\ &= A(x,\theta_m)\cos\left[f_m(\theta_m) \pm \frac{\partial f_m(\theta_m)}{\partial\lambda'}\Delta\lambda' \pm \frac{\partial f_m(\theta_m)}{\partial\theta_m}\Delta\theta_m\right] \\ &= A(x,\theta_m)\cos\left[\frac{(m+\frac{1}{2})}{2}\left(\frac{\sin(2\theta_m)}{2} - \theta_m\right)\right. \\ &\qquad \left.\pm\frac{1}{2}\left(\frac{\sin(2\theta_m)}{2} - \theta_m\right) \mp \frac{1}{4}\sin 2\theta_m - (-1)^m\frac{\chi}{4} + \frac{\pi}{4}\right].\end{aligned} \tag{A.18}$$

Thus we have

$$\begin{aligned}&\phi^{(2)}_{m\pm1}(x) \\ &\quad = \frac{2C_m\sqrt{x}}{\sqrt{\sin\theta_m}}\cos\left[\frac{(m+\frac{1}{2})}{2}\left(\frac{\sin 2\theta_m}{2} - \theta_m\right) \mp \frac{\theta_m}{2} - (-1)^m\frac{\chi}{4} + \frac{\pi}{4}\right].\end{aligned} \tag{A.19}$$

Here we note that for a given x, χ remains constant under the variation of θ and m. $\square$

APPENDIX B

Associated Laguerre and Gaussian Results as Limiting Cases of Jacobi Skew-orthogonal Polynomials

In this appendix we prove the Jacobi results (8.6) – (8.15) and the associated Laguerre and Gaussian results for the skew-orthogonal polynomials given in Chapters 6, 8 for $\beta = 1, 4$ respectively by taking the limit.

For associated Laguerre and Hermite weight functions one can directly follow the above procedure, or more simply take the limits of the Jacobi results as discussed below. For associated Laguerre, note first that

$$w_a(x) = \lim_{b\to\infty} 2^{-a-b} b^a w_{ab}(1 - 2b^{-1}x), \tag{B.1}$$

$$L_j^{(a)}(x) = \lim_{b\to\infty} P_j^{a,b}(1 - 2b^{-1}x), \tag{B.2}$$

$$k_j^{(a)} = \lim_{b\to\infty} \left(-\frac{2}{b}\right)^j k_j^{a,b}, \tag{B.3}$$

$$h_j^{(a)} = \lim_{b\to\infty} \frac{b^{a+1}}{2^{a+b+1}} h_j^{a,b}. \tag{B.4}$$

Thus for skew-orthogonal functions we have (in terms of the Jacobi skew-orthogonal functions $\phi_j^{a,b}$, $\psi_j^{a,b}$),

$$\phi_j^{(1)}(x) = \lim_{b\to\infty} (-1)^j 2^{-b+1/2} b^a \phi_j^{a,b}(1 - 2b^{-1}x), \tag{B.5}$$

$$\psi_j^{(1)}(x) = \lim_{b\to\infty} (-1)^{j-1} 2^{-b-1/2} b^{a+1} \psi_j^{a,b}(1 - 2b^{-1}x), \tag{B.6}$$

giving thereby (4.40) – (4.50). Similarly, for the Hermite weight, note that (with $j = 2m, 2m+1$)

$$\mathrm{e}^{-x^2/2} = \lim_{a\to\infty} w_{a,a}\left(\frac{x}{\sqrt{2a}}\right), \tag{B.7}$$

$$H_j(x) = \lim_{a\to\infty} 2^j j! a^{-j/2} P_j^{a,a}\left(\frac{x}{\sqrt{a}}\right), \tag{B.8}$$

$$k_j = \lim_{a\to\infty} 2^j j! a^{-j} k_j^{a,a}, \tag{B.9}$$

$$h_j = \lim_{a\to\infty} (2^j j!)^2 a^{-j+1/2} h_j^{a,a}, \tag{B.10}$$

$$\phi_j^{(1)}(x) = \lim_{a\to\infty} 2^{2m} (2m)! (2a)^{-j/2} \phi_j^{a,a}\left(\frac{x}{\sqrt{2a}}\right), \tag{B.11}$$

$$\psi_j^{(1)}(x) = \lim_{a\to\infty} 2^{2m} (2m)! (2a)^{-(j-1)/2} \psi_j^{a,a}\left(\frac{x}{\sqrt{2a}}\right), \tag{B.12}$$

giving the Hermite results (4.52) – (4.57).

For $\beta = 4$ with the Jacobi weight function, we expand $\pi_j^{(4)\prime}(x)$ as

$$\pi_j^{(4)\prime}(x) = P_{j-1}^{a,b}(x) + \sum_{k=0}^{j-1} \eta_k^{(j)} \pi_k^{(4)\prime}(x), \tag{B.13}$$

so that (8.10) gives

$$\pi_j^{(4)}(x) = \frac{2}{j+a+b-1}\big[D_j P_j^{a,b}(x) + E_j P_{j-1}^{a,b}(x) + F_j P_{j-2}^{a,b}(x)\big] + \sum_{k=0}^{j-1} \eta_k^{(j)} \pi_k^{(4)}(x). \tag{B.14}$$

Then the orthogonality of the $P_j^{a,b}(x)$ and the skew-orthogonality of the $\pi_j^{(4)}(x)$ give

$$\eta_k^{(2m)} = 0, \tag{B.15}$$

for $k \neq 2m-2$, and

$$\eta_k^{(2m+1)} = 0 \tag{B.16}$$

for $k \neq 2m$. Also $\eta_{2m}^{(2m+1)}$, being arbitrary, is chosen to be zero. Thus $\eta_{2m-2}^{(2m)} \equiv \eta_{2m}$ is the only nonzero coefficient, giving thereby (8.6) – (8.9). The skew-orthogonality of $\pi_{2m}^{(4)}(x)$ and $\pi_{2m-1}^{(4)}(x)$ gives

$$\eta_{2m} = \frac{1}{g_{2m-2}}\left[\frac{2D_{2m-1}}{2m+a+b-2} h_{2m-1}^{a,b} - \frac{2F_{2m}}{2m+a+b-1} h_{2m-2}^{a,b}\right], \tag{B.17}$$

while the normalization is given by

$$g_{2m}^{(4)} = \left[\frac{2D_{2m}}{2m+a+b-1} h_{2m}^{a,b} - \frac{2F_{2m+1}}{2m+a+b} h_{2m-1}^{a,b}\right], \tag{B.18}$$

confirming thereby (8.14), (8.15). To prove the Jacobi result (8.10), we note that the first step is given in a differential form in [77], while for the second step we use [1]

$$(2j+a+b) P_j^{a,b-1}(x) = (j+a+b) P_j^{a,b}(x) + (j+a) P_{j-1}^{a,b}(x), \tag{B.19}$$

$$(2j+a+b) P_j^{a-1,b}(x) = (j+a+b) P_j^{a,b}(x) - (j+b) P_{j-1}^{a,b}(x). \tag{B.20}$$

This completes the proof of (8.6) – (8.15).

The associated Laguerre results (8.37) – (8.42) derive directly by using the limits (B.1) – (B.4) in (8.6) – (8.15), while the Hermite results (8.64) – (8.67) derive from the limits

$$\mathrm{e}^{-2x^2} = \lim_{a\to\infty} w_{a,a}(x\sqrt{2/a}), \tag{B.21}$$

$$H_j(x\sqrt{2}) = \lim_{a\to\infty} 2^j j! a^{-j/2} P_j^{a,a}(x\sqrt{2/a}). \tag{B.22}$$

APPENDIX C

Proofs of (10.2) – (10.9)

In this appendix we outline a proof of the matrix-integral representations (10.2) – (10.9) of the polynomials. The Vandermonde determinant and its fourth power can be written as

$$\Delta(x_1,\ldots,x_N)=\det[x_\mu^{N-\nu}]_{\mu,\nu=1,\ldots,N}, \tag{C.1}$$

$$(\Delta(x_1,\ldots,x_N))^4=\det[x_\mu^{2N-\nu},(2N-\nu)x_\mu^{2N-\nu-1}]_{\mu=1,\ldots,N,\nu=1,\ldots,2N}. \tag{C.2}$$

For $\beta=2$, (10.2) represents orthogonal polynomials with the weight $w(x)$ if

$$\int x^k P_j(x)w(x)\ \mathrm{d}x=0,\qquad k=0,1,\ldots,j-1. \tag{C.3}$$

Using the joint-probability distribution (1.12) result, in the definition of average (10.1) and using the determinant in (10.2), the integral in (C.3) is proportional to

$$\begin{aligned}\int \mathrm{d}x_1\cdots\int \mathrm{d}x_{j+1}\,(x_{j+1})^k\Delta(x_1,\ldots,x_j)\Delta(x_1,\ldots,x_{j+1})\prod_{\mu=1}^{j+1}w(x_\mu)\\ =\frac{1}{(j+1)!}\int \mathrm{d}x_1\cdots\int \mathrm{d}x_{j+1}\left(\sum_P\epsilon_P(x_{i_{j+1}})^k\Delta(x_{i_1},\ldots,x_{i_j})\right)\\ \times\Delta(x_1,\ldots,x_{j+1})\prod_{\mu=1}^{j+1}w(x_\mu),\end{aligned} \tag{C.4}$$

where $\sum_P$ is summation over all permutations $(x_{i_1},\ldots,x_{i_{j+1}})$ of $(x_1,\ldots,x_{j+1})$ and ϵ_P $(=\pm1)$ is the sign of the permutation, equal to the change of sign in $\Delta(x_1,\ldots,x_{j+1})$ after the permutation. The summation term in (C.4) can be written as

$$\sum_P\epsilon_P(x_{i_{j+1}})^k\Delta(x_{i_1},\ldots,x_{i_j})=(-1)^j\det\begin{pmatrix}x_1^k & x_2^k & \ldots & x_{j+1}^k\\ x_1^{j-1} & x_2^{j-1} & \ldots & x_{j+1}^{j-1}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & \ldots & 1\end{pmatrix}, \tag{C.5}$$

which is zero for $k=0,1,\ldots,j-1$, thereby proving (C.3) and hence (10.2).

For $\beta=1$, we consider (10.3), (10.4) for the even-N case; a similar consideration would apply to (10.5) – (10.7) for the odd-N case. The $\pi_j^{(1)}(x)$ of (10.3), (10.4) represent skew-orthogonal polynomials of the $\beta=1$ type with the weight $w(x)$ if

$$\iint \mathrm{d}x\ \mathrm{d}y\ \epsilon(x-y)y^k\pi_j^{(1)}(x)w(x)w(y)=0, \tag{C.6}$$

for $k=0,\ldots,2m-1$ and also for $k=j$ for both $j=2m,2m+1$. The integrals in (10.3), (10.4) involve $|\Delta(x_1,\ldots,x_{2m})|$ and therefore Mehta's method of integration over alternate variables [53] can be used. For $j = 2m$, the integral in (C.6) is proportional to

$$
\begin{aligned}
\text{(C.7)}\quad &\int \mathrm{d}x_1 \cdots \mathrm{d}x_{2m+2}\ \epsilon(x_{2m+1} - x_{2m+2})(x_{2m+2})^k \\
&\qquad\qquad \times \left(\prod_{\nu=1}^{2m}(x_{2m+1} - x_\nu)\right)|\Delta(x_1, \dots, x_{2m})| \prod_{\mu=1}^{2m+2} w(x_\mu) \\
&= (2m)! \int_{x_1 \le x_2 \le \cdots \le x_{2m}} \mathrm{d}x_1 \cdots \int \mathrm{d}x_{2m+2}\ \epsilon(x_{2m+1} - x_{2m+2})(x_{2m+2})^k \\
&\qquad\qquad \times \left(\prod_{\mu=1}^{2m+2} w(x_\mu)\right)\Delta(x_1, \dots, x_{2m+1}) \\
&= \frac{1}{2}\frac{(2m)!}{m!} \int \mathrm{d}x_1\ \mathrm{d}x_3 \cdots \mathrm{d}x_{2m+1}\ \det\begin{pmatrix} 0 & 0 & \dots & x_{2m+1}^k & F_k(x_{2m+1}) \\ x_1^{2m} & F_{2m}(x_1) & \dots & x_{2m}^{2m} & F_{2m}(x_{2m+1}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & F_0(x_1) & \dots & 1 & F_0(x_{2m+1}) \end{pmatrix} \\
&\qquad\qquad \times \prod_{i=0}^{m} w(x_{2i+1}) \\
&= \frac{1}{2}\frac{(2m)!}{m!(m+1)!} \int \mathrm{d}x_1\ \mathrm{d}x_3 \cdots \mathrm{d}x_{2m+1}\ \det\begin{pmatrix} x_1^k & F_k(x_1) & \dots \\ x_1^{2m} & F_{2m}(x_1) & \dots \\ x_1^{2m-1} & F_{2m-1}(x_1) & \dots \\ \vdots & \vdots & \vdots \\ 1 & F_0(x_1) & \dots \end{pmatrix} \\
&\qquad\qquad \times \prod_{i=0}^{m} w(x_{2i+1}).
\end{aligned}
$$

In the second and third steps the above-mentioned Mehta's method of integration over alternate variables is used, where $F_k(x)$ is given by

$$
\text{(C.8)}\qquad F_k(x) = \int_x^\infty y^k w(y)\ \mathrm{d}y.
$$

In the last step of (C.7) all permutations of $(x_1, x_3, \dots, x_{2m+1})$ have been used. The determinant in the last step is zero for $k = 0, \dots, 2m$, thereby proving (C.6) and (10.3). For $j = 2m + 1$, the first integral in (C.7) has, in the integrand, the extra factor $(x_{2m+1} + \sum x_\nu)$ so that $\Delta(x_1, \dots, x_{2m+1})$ in the second form is replaced by

$$
\text{(C.9)}\qquad \det\begin{pmatrix} x_1^{2m+1} & \dots & x_{2m+1}^{2m+1} \\ x_1^{2m-1} & \dots & x_{2m+1}^{2m-1} \\ \vdots & \dots & \vdots \\ 1 & \dots & 1 \end{pmatrix} = \left(\sum_{\mu=1}^{2m+1} x_\mu\right)\Delta(x_1, \dots, x_{2m+1}).
$$

Then the second row in the determinant of the last step (C.7) is replaced by $x_1^{2m+1}, F_{2m+1}(x_1), \dots,$ other rows remaining the same. Again (C.6) for $k = 0, \dots, 2m-1, 2m+1$ and hence (10.4) are verified.

For $\beta = 4$, (10.8) – (10.9) represent the skew-orthogonal polynomials if

$$\int \mathrm{d}x\, \{x^k \pi_j^{(4)\prime}(x) - kx^{k-1}\pi_j^{(4)}(x)\} w(x) = 0, \tag{C.10}$$

for $k = 0, \ldots, 2m-1$ and also for $k = j$ for $j = 2m$, $2m+1$ both. In this case we use (C.2) in the joint-probability density (1.12). For $j = 2m$, the integral in (C.10) is proportional to

$$\begin{aligned}\int \mathrm{d}x_1 \cdots \mathrm{d}x_{2m}\, \mathrm{d}x_{2m+1} \\ \left\{x_{2m+1}^k \frac{\mathrm{d}}{\mathrm{d}x_{2m+1}} \prod_{\nu=1}^{m} (x_{2m+1} - x_\nu)^2 - kx_{2m+1}^{k-1} \prod_{\nu=1}^{m} (x_{2m+1} - x_\nu)^2\right\} \\ \times \left(\prod_{\mu=1}^{2m+1} w(x_\mu)\right) \big(\Delta(x_1, \ldots, x_m)\big)^4 \\ = \int \mathrm{d}x_1 \cdots \mathrm{d}x_{2m+1}\, \det \begin{pmatrix} 0 & 0 & \ldots & x_{2m+1}^k & kx_{2m+1}^{k-1} \\ x_1^{2m} & 2mx_1^{2m-1} & \ldots & x_{2m+1}^{2m} & 2mx_{2m+1}^{2m-1} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 1 & 0 & \ldots & 1 & 0 \end{pmatrix} \prod_{\mu=1}^{2m+1} w(x_\mu) \\ = \frac{1}{(m+1)!} \int \mathrm{d}x_1 \cdots \mathrm{d}x_{2m+1}\, \det \begin{pmatrix} x_1^k & kx_1^{k-1} & \ldots \\ x_1^{2m} & 2mx_1^{2m-1} & \ldots \\ \vdots & \vdots & \vdots \\ 1 & 0 & \ldots \end{pmatrix} \prod_{\mu=1}^{2m+1} w(x_\mu),\end{aligned} \tag{C.11}$$

where the last step is by a permutation of all the variables in the first step. The determinant in the last step is again zero for $k = 0, \ldots, 2m$, confirming (C.10) and hence (10.8). For $j = 2m+1$, we have the additional term $(x_{2m+1} + 2\sum x_\nu)$ with $\prod (x - x_\nu)^2$. In this case the second rows of both the determinants of (C.11) are replaced by $(x_1^{2m+1}, (2m+1)x_1^{2m}, \ldots)$, the last determinant being then zero for $k = 0, \ldots, 2m-1$, and $2m+1$. Thus (10.9) is verified.

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