

Volume 30

CRM
CRM
M

CRM
MONOGRAPH
SERIES

Centre de Recherches Mathématiques
Montréal

Moduli Spaces and
Arithmetic Dynamics

Joseph H. Silverman



American Mathematical Society

Moduli Spaces and Arithmetic Dynamics

Volume 30



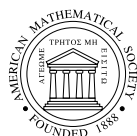
CRM MONOGRAPH SERIES

Centre de Recherches Mathématiques
Montréal

Moduli Spaces and Arithmetic Dynamics

Joseph H. Silverman

The Centre de Recherches Mathématiques (CRM) of the Université de Montréal was created in 1968 to promote research in pure and applied mathematics and related disciplines. Among its activities are special theme years, summer schools, workshops, postdoctoral programs, and publishing. The CRM is supported by the Université de Montréal, the Province of Québec (FQRNT), and the Natural Sciences and Engineering Research Council of Canada. It is affiliated with the Institut des Sciences Mathématiques (ISM) of Montréal. The CRM may be reached on the Web at www.crm.math.ca.



American Mathematical Society
Providence, Rhode Island USA

The author's research was supported by NSF DMS-0650017 and DMS-0854755.

2000 *Mathematics Subject Classification*. Primary 37P45; Secondary 37A45, 37F45, 37P30, 37PXX, 14D20, 14D22.

Library of Congress Cataloging-in-Publication Data

Silverman, Joseph H.

Moduli spaces and arithmetic dynamics / Joseph H. Silverman.

p. cm. — (CRM monograph series, Centre de recherches mathématiques, Montréal : v. 30)

Includes bibliographical references and index.

ISBN 978-0-8218-7582-7 (alk. paper)

1. Moduli theory. 2. Analytic spaces. 3. Ergodic theory. 4. Harmonic analysis. I. Title.

QA331.S519 2012

516.3'5—dc23

2011046247

Copying and reprinting. Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy a chapter for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication is permitted only under license from the American Mathematical Society. Requests for such permission should be addressed to the Acquisitions Department, American Mathematical Society, 201 Charles Street, Providence, Rhode Island 02904-2294 USA. Requests can also be made by e-mail to reprint-permission@ams.org.

© 2012 by the American Mathematical Society. All rights reserved.

The American Mathematical Society retains all rights
except those granted to the United States Government.

Printed in the United States of America.

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.

This volume was submitted to the American Mathematical Society
in camera ready form by the Centre de Recherches Mathématiques.

Visit the AMS home page at <http://www.ams.org/>

10 9 8 7 6 5 4 3 2 1 17 16 15 14 13 12

Contents

| | |
|---------------------------------------------------------------|-----|
| Preface | vii |
| Introduction | 1 |
| Chapter 1. Moduli Spaces Associated to Dynamical Systems | 3 |
| 1.1. Dynamical definitions | 3 |
| 1.2. Moduli spaces: what they are and why they're useful | 5 |
| 1.3. Fine moduli spaces and coarse moduli spaces | 5 |
| 1.4. Parameter spaces for dynamical systems | 6 |
| 1.5. Moduli spaces for dynamical systems | 8 |
| 1.6. Level structure and the uniform boundedness conjecture | 8 |
| Chapter 2. The Geometry of Dynamical Moduli Spaces | 11 |
| 2.1. Introduction to geometric invariant theory (GIT) | 12 |
| 2.2. Tools for computing the stable and semistable loci | 17 |
| 2.3. Construction of moduli spaces M_d^n using GIT | 20 |
| 2.4. Multipliers and maps on M_d | 24 |
| 2.5. M_2 is isomorphic to \mathbb{A}^2 | 30 |
| 2.6. Uniform bounds for $\text{Aut}(\phi)$ | 34 |
| 2.7. Rationality of M_d^1 | 36 |
| 2.8. Special loci in M_d^n | 39 |
| Chapter 3. Dynamical Moduli Spaces—Further Topics | 45 |
| 3.1. An application to good reduction over function fields | 45 |
| 3.2. Minimal resultants and minimal models | 47 |
| 3.3. Dynamics on K3 Surfaces | 50 |
| 3.4. An algebraic characterization of Lattès maps | 53 |
| Chapter 4. Dynatomic Polynomials and Dynamical Modular Curves | 57 |
| 4.1. Dynatomic polynomials | 57 |
| 4.2. Dynamical modular curves for $z^2 + c$ | 59 |
| 4.3. Irreducibility and genus formulas | 60 |
| 4.4. Rational points on dynamical modular curves | 65 |
| 4.5. Other dynamical modular curves | 65 |
| Chapter 5. Canonical Heights | 69 |
| 5.1. Heights and projective space | 69 |
| 5.2. Dynamical canonical heights | 70 |
| 5.3. Canonical height zero over function fields | 73 |
| 5.4. Local heights and Green functions | 76 |
| 5.5. Specialization theorems | 77 |

| | |
|---------------------------------------------------------|-----|
| 5.6. Heights and dominant rational maps | 81 |
| 5.7. Canonical heights for regular affine automorphisms | 83 |
| 5.8. Canonical heights for K3 dynamics | 84 |
| 5.9. Algebraic dynamics and transcendental numbers | 88 |
| Chapter 6. Postcritically Finite Maps | 91 |
| 6.1. Transversality of the PCF locus | 92 |
| 6.2. The height of a postcritically finite map | 100 |
| 6.3. The invariant measure and the Lyapunov exponent | 107 |
| 6.4. Conservative maps | 109 |
| 6.5. A dynamical André–Oort conjecture | 110 |
| Chapter 7. Field of Moduli and Field of Definition | 113 |
| 7.1. Twists, automorphisms, and cohomology | 113 |
| 7.2. Fields of definition and field of moduli | 116 |
| 7.3. Tools for determining when FOM = FOD | 117 |
| Schedule of Talks at the Bellairs Workshop | 121 |
| Glossary | 123 |
| Bibliography | 127 |
| Index | 133 |

Preface

This monograph is an expanded version of the notes for a series of lectures delivered at a workshop on *Moduli Spaces and the Arithmetic of Dynamical Systems*, Bellairs Research Institute, Barbados, May 2–9, 2010. As such, the level of exposition is uneven, with some results being worked out in detail, while others are merely sketched or have proofs by citation. The goal is to provide an overview, with enough details and pointers to the existing literature, to give the reader an entry into this exciting area of current research. It is the author’s hope that this will be useful, especially since at present there are only a small number of books [4, 38, 61, 90, 99, 110] dealing with the arithmetic or algebraic side of dynamical systems. For further reading, the reader might consult the webpage

<http://www.math.brown.edu/~jhs/ADSHome.html>

which contains links to an extensive list of articles in this area.

Acknowledgements. I would like to thank Xander Faber and the McGill University Mathematics Department for inviting me to deliver a series of lectures at the Bellairs Workshop, and Chantal David and Ina Mette for arranging for these notes to be published as a CRM monograph.

I would like to thank Shu Kawaguchi for showing me the proof of Proposition 3.18, which he adapted from [52, Proposition 21.6], Michelle Manes for providing the proof sketch of Bousch’s theorem (Theorem 4.12), Michelle Manes and Alon Levy for the content of Remark 4.19, Tom Scanlon for providing the proof sketch of Theorem 5.11(b), Adam Epstein for providing the proof sketch of Proposition 6.18, Michael Zieve for allowing me to include the algebraic characterization of Lattès maps (Theorem 3.22), Curt McMullen for providing information about transcendence in dynamics, and Xavier Buff, Laura DeMarco, and Adam Epstein for a number of helpful email conversations.

I would also like to thank the people who looked at an initial draft of this monograph and offered suggestions and corrections: Adam Epstein, Ben Hutz, Patrick Ingram, Michelle Manes, Bjorn Poonen.

These notes greatly benefited from the many questions and comments posed by the participants at the Bellairs Workshop, so I would like to thank all of them for being such a lively audience: Arthur Baragar, Anupam Bhatnagar, Henri Darmon, Laura DeMarco, Adam Epstein, Xander Faber, Eyal Goren, Benjamin Hutz, Patrick Ingram, Rafe Jones, Shu Kawaguchi, Cristin Kenney, Sarah Koch, Holly Krieger, ChongGyu (Joey) Lee, Alon Levy, Kalyani Madhu, Michelle Manes, Alice Medvedev, Bjorn Poonen, Michael L. Tepper, Adam Towsley, and Phillip Williams.

Joseph H. Silverman
August 2011

Schedule of Talks at the Bellairs Workshop

- **Monday:**
 - Joseph Silverman: *Introduction to Moduli Spaces*
 - Laura DeMarco: *Moduli Spaces of One-Dimensional Dynamical Systems*
- **Tuesday:**
 - Joseph Silverman: *Dynamical Moduli Spaces*
 - Adam Epstein: *Transversality and Holomorphic Dynamics*
- **Wednesday:**
 - Joseph Silverman: *Dynatomic Polynomials and Dynamical Modular Curves*
 - Michelle Manes: *Level Structures on Dynamical Moduli Spaces*
- **Thursday:**
 - Joseph Silverman: *Canonical Heights*
 - * Anupam Bhatnagar: *Points of Canonical Height Zero on Projective Varieties*
 - * ChongGyu (Joey) Lee: *Height Estimates for Dominant Morphisms*
 - * Alon Levy: *Semistable Reduction for Dynamical Systems*
- **Friday:**
 - Joseph Silverman: *Field of Moduli and Fields of Definition*
 - Michael Tepper: *Isotriviality and M_d^n*

Glossary

| | |
|-------------------------------|----------------------------------------------------------------------------------|
| $[F_n, G_n]$ | n th iterate of $[F, G]$, 57 |
| $[\phi, \psi]_{AZ}$ | Arakelov–Zhang pairing, 72 |
| $[\phi, \psi]_{KS}$ | Kawaguchi–Silverman pairing, 72 |
| $F \simeq G$ | notation for $F \ll G$ and $G \ll F$, 101 |
| $g \cdot x$ | the image of the action of g on x , 12 |
| $\langle \phi \rangle$ | the point in M_d^n corresponding to $\phi \in \text{Hom}_d^n$, 25 |
| $G \ll F$ | notation for $G(x) \leq c_1 F(x) + c_2$, 101 |
| | |
| A^G | the ring of invariant of G acting on A , 11 |
| A_g | the moduli space of principally polarized abelian varieties, 88 |
| $\bar{\mathbf{a}}(\phi)$ | ideal class associated to minimal model of ϕ , 49 |
| $\text{Aut}(\phi)$ | the automorphism group of the map ϕ , 34 |
| | |
| BiCrit_d | set of maps with exactly two critical points, 42 |
| | |
| $C_1(n)$ | dynamical modular curve for maps $\psi_b(z) = z/(z^2 + b)$, 66 |
| | |
| $\deg_X(S)$ | minimal degree of polynomial vanishing on S , 98 |
| | |
| $e_\phi(P)$ | the ramification index of ϕ at P , 91 |
| $e_v(\phi)$ | the valuation of the Macaulay resultant of ϕ , 47 |
| $\varepsilon_v(\phi)$ | exponent at v of the minimal resultant of ϕ , 48 |
| | |
| F_g | moduli space of K3 surfaces, 52 |
| Fix | map from Hom_d to the fixed points of the map, 36 |
| $\Phi_{\phi, n}$ | n -dynatonic polynomial, 57 |
| ϕ^f | $= f^{-1} \circ \phi \circ f$, the conjugate of ϕ by f , 4 |
| ϕ^n | n th iterate of the function ϕ , 3 |
| $\phi^\#(P)$ | chordal derivative of ϕ at P , 108 |
| $\phi_c(z)$ | the polynomial $z^2 + c$, 59 |
| | |
| G_ϕ | the subgroup of $\text{Gal}(\bar{K}/K)$ such that $\sigma(\phi) \sim \phi$, 116 |
| \mathbb{G}_m | the multiplicative group, 15 |
| $\widehat{G}_{\Phi, v}(P)$ | Green function for lift Φ of ϕ , 76 |
| $\hat{g}_{\phi, v}$ | Green function (local canonical height), 76 |
| | |
| $H^1(G, A)$ | cohomology set for G acting on A , 115 |
| $\hat{h}_{\text{crit}}(\phi)$ | critical canonical height of the map ϕ , 99 |

| | |
|------------------------------------------|-----------------------------------------------------------------------------------------------------|
| $\hat{h}_\phi^+, \hat{h}_\phi^-$ | canonical heights for a regular affine automorphism, 83 |
| $\hat{h}_\phi^+, \hat{h}_\phi^-$ | canonical heights on a K3 surface, 86 |
| h | the Weil height on $\mathbb{P}^N(\overline{K})$, 69 |
| $h(\phi)$ | the height of the map ϕ via $\phi \in \text{Hom}_d^N \subset \mathbb{P}^N$, 70 |
| $(\text{Hom}_d)^{\text{ss}}$ | semistable locus in Hom_d , 21 |
| $(\text{Hom}_d)^{\text{s}}$ | stable locus in Hom_d , 21 |
| $\text{Hom}_d(\mathbb{P}^1(\mathbb{C}))$ | degree d rational self-maps of $\mathbb{P}^1(\mathbb{C})$, 3 |
| Hom_d^n | degree d morphisms $\mathbb{P}^n \rightarrow \mathbb{P}^n$, 7 |
| $\text{Hom}_d^n(m)$ | maps with a marked point of formal period n , 8 |
| ι_1, ι_2 | involutions on the surface $S_{\mathbf{A}, \mathbf{B}}$, 51 |
| $\mathcal{J}^f(\phi)$ | the filled Julia set of ϕ , 76 |
| K_ϕ | the field of moduli for ϕ , 116 |
| $L(\phi)$ | the Lyapunov exponent of ϕ , 108 |
| \mathbf{L}_d | the set of flexible Lattès maps in \mathbf{M}_d , 101 |
| $\ell(0) \cdot x$ | specialization of the 1-parameter subgroup ℓ , 17 |
| Λ_ϕ^n | the multiplier spectrum of ϕ , 25 |
| $\lambda_\phi(\alpha)$ | multiplier of ϕ at the periodic point α , 24 |
| Lat_d | the set of flexible Lattès maps in Hom_d , 101 |
| $(\mathbf{M}_d)^{\text{ss}}$ | moduli space of semistable points in Hom_d , 21 |
| $(\mathbf{M}_d)^{\text{s}}$ | moduli space of stable points in Hom_d , 21 |
| M_K | complete set of inequivalent normalized absolute values on K , 69 |
| M_K | set of absolute values on the function field $K = k(C)$, 45 |
| \mathbf{M}_d | moduli space of self-morphisms of \mathbb{P}^1 , 8 |
| \mathbf{M}_d^n | moduli space of self-morphisms of \mathbb{P}^n , 8 |
| $\mathbf{M}_d^n(m)$ | moduli space of maps with a marked point of formal period n , 8 |
| $\mathbf{M}_d^{\text{BiCrit}}$ | image of BiCrit_d in \mathbf{M}_d , 43 |
| $\mathbf{M}_d^{\text{crit}}$ | moduli space of degree d maps with marked critical points, 92 |
| $\mathbf{M}_d^{\text{crit}}[i](r, n)$ | subvariety of $\mathbf{M}_d^{\text{crit}}$ with marked critical point having specified portrait, 93 |
| \mathfrak{M} | the Mandelbrot set, 88 |
| $\overline{\mathbf{M}}_2$ | completion of \mathbf{M}_2 , 33 |
| $\hat{\mu}_\phi$ | invariant measure associated to ϕ , 107 |
| $\mu(\phi)$ | height expansion coefficient of ϕ , 81 |
| $\mu^{\mathcal{L}}(x, \ell)$ | integer invariant attached to 1-parameter subgroup ℓ , 18 |
| $\mathcal{O}_\phi(x)$ | the forward orbit of x for the map ϕ , 3 |
| $\mathcal{O}_{\phi, \psi}(P)$ | full orbit of P for two maps ϕ and ψ , 80 |
| $\mathbf{P}_d^{\text{crit}}$ | subvariety of $\mathbf{M}_d^{\text{crit}}$ corresponding to polynomial maps, 96 |

| | |
|-----------------------------------|------------------------------------------------------------------------------------|
| $P_d^{\text{crit}}[i](r, n)$ | subvariety of P_d^{crit} with point having given critical point orbit, 96 |
| $\text{Per}(\phi, X)$ | set of periodic points for ϕ , 4 |
| $\text{Per}_n(\phi, X)$ | set of periodic points of period n for ϕ , 4 |
| $\text{Per}_n^*(\phi)$ | periodic points of formal period n , 92 |
| $\text{Per}_n^{**}(\phi)$ | points of exact period n , 8 |
| PGL_{n+1} | the projective linear group, 8 |
| $\text{PrePer}(\phi, X)$ | set of preperiodic points for ϕ , 4 |
| $\text{PrePer}_{m,n}(\phi, X)$ | set of preperiodic points of tail m and period n , 4 |
| $\text{PrePer}_{r,n}^*(\phi)$ | preperiodic points of tail length r and formal period n , 92 |
| $\mathfrak{R}(\phi)$ | the minimal resultant of ϕ , 48 |
| Rat_d^n | degree d rational maps $\mathbb{P}^n \rightarrow \mathbb{P}^n$, 7 |
| $(\mathcal{S}\phi)(z)$ | the Schwarzian derivative of ϕ , 29 |
| $S_{\mathbf{A}, \mathbf{B}}$ | K3 surface determined by the coefficients \mathbf{A} and \mathbf{B} , 50 |
| \mathcal{S}_n | symmetric group, 11 |
| $\sigma(g, x)$ | the image of the action of g on x , 12 |
| $\sigma_{i,n}(\phi)$ | symmetric function of multipliers of ϕ , 25 |
| $\text{Stab}(f)$ | the stabilizer of the map f , 12 |
| $\mathcal{T}_P(\mathbb{P}^1)$ | tangent space of \mathbb{P}^1 at P , 25 |
| $\mathcal{T}_P(\mathbb{P}^N)$ | tangent space of \mathbb{P}^N at P , 27 |
| $\text{Twist}_K(\phi)$ | the set of K -twists of ϕ , 113 |
| V/G | the quotient of V by the finite group G , 11 |
| $X^{\text{ss}}(\mathcal{L})$ | semistable locus, 16 |
| $X^{\text{s}}(\mathcal{L})$ | stable locus, 16 |
| $X_0(n)$ | smooth projective model for $Y_0(n)$, 60 |
| $X_1(n)$ | smooth projective model for $Y_1(n)$, 59 |
| $X_{(0)}^{\text{s}}(\mathcal{L})$ | stable locus with dimension 0 stabilizer, 16 |
| $Y_0(n)$ | dynamical modular curve, 60 |
| $Y_1(n)$ | dynamical modular curve, 59 |
| $Z(\phi)$ | locus of indeterminacy of ϕ , 40 |

Bibliography

1. Y. André, *G-functions and geometry*, Aspects Math., vol. E13, Friedr. Vieweg, Braunschweig, 1989.
2. M. Baker, *A finiteness theorem for canonical heights attached to rational maps over function fields*, J. Reine Angew. Math. **626** (2009), 205–233.
3. M. Baker and L. DeMarco, *Preperiodic points and unlikely intersections*, Duke Math. J. **159** (2011), no. 1, 1–29.
4. M. Baker and R. Rumely, *Potential theory and dynamics on the Berkovich projective line*, Math. Surveys Monogr., vol. 159, Amer. Math. Soc., Providence, RI, 2010.
5. A. Baragar, *Rational points on K3 surfaces in $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$* , Math. Ann. **305** (1996), no. 3, 541–558.
6. ———, *Rational curves on K3 surfaces in $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$* , Proc. Amer. Math. Soc. **126** (1998), no. 3, 637–644.
7. ———, *Canonical vector heights on algebraic K3 surfaces with Picard number two*, Canad. Math. Bull. **46** (2003), no. 4, 495–508.
8. ———, *Orbits of curves on certain K3 surfaces*, Compositio Math. **137** (2003), no. 2, 115–134.
9. ———, *Canonical vector heights on K3 surfaces with Picard number three—an argument for nonexistence*, Math. Comp. **73** (2004), no. 248, 2019–2025 (electronic).
10. A. Baragar and R. van Luijk, *K3 surfaces with Picard number three and canonical vector heights*, Math. Comp. **76** (2007), no. 259, 1493–1498 (electronic).
11. P.-G. Becker, *Transcendence of the values of functions satisfying generalized Mahler type functional equations*, J. Reine Angew. Math. **440** (1993), 111–128.
12. P.-G. Becker and W. Bergweiler, *Transcendence of local conjugacies in complex dynamics and transcendence of their values*, Manuscripta Math. **81** (1993), no. 3-4, 329–337.
13. R. L. Benedetto, *Heights and preperiodic points of polynomials over function fields*, Int. Math. Res. Not. **62** (2005), 3855–3866.
14. D. Bertrand, *Problèmes de transcendance liés aux hauteurs sur les courbes elliptiques*, Mathématiques, CTHS: Bull. Sec. Sci., vol. 3, Bib. Nat., Paris, 1981, pp. 55–63.
15. A. Bhatnagar, *Points of canonical height zero on projective varieties* (2010).
16. A. Bhatnagar and L. Szpiro, *Canonical height zero over function fields*, in preparation.
17. E. Bombieri and W. Gubler, *Heights in Diophantine geometry*, New Math. Monogr., vol. 4, Cambridge Univ. Press, Cambridge, 2006.
18. E. Bombieri, D. Masser, and U. Zannier, *Intersecting a curve with algebraic subgroups of multiplicative groups*, Internat. Math. Res. Notices **20** (1999), 1119–1140.
19. ———, *Anomalous subvarieties—structure theorems and applications*, Int. Math. Res. Not. IMRN **19** (2007), Art. ID rnm057, 33 pp.
20. ———, *On unlikely intersections of complex varieties with tori*, Acta Arith. **133** (2008), no. 4, 309–323.
21. T. Bousch, *Sur quelques problèmes de dynamique holomorphe*, Ph.D. Thesis, Université Paris-Sud 11, 1992.
22. J.-Y. Briend and J. Duval, *Exposants de Liapounoff et distribution des points périodiques d’un endomorphisme de \mathbf{CP}^k* , Acta Math. **182** (1999), no. 2, 143–157.
23. X. Buff and A. Epstein, *Bifurcation measure and postcritically finite rational maps*, Complex Dynamics (D. Schleicher, ed.), A K Peters, Wellesley, MA, 2009, pp. 491–512.
24. X. Buff and L. Tan, *The quadratic dynatomic curves are smooth and irreducible*, preprint.
25. L. Busé, *Elimination theory in codimension one and applications*, Technical Report 5918, INRIA, 2006.

26. G. S. Call and J. H. Silverman, *Canonical heights on varieties with morphisms*, *Compositio Math.* **89** (1993), no. 2, 163–205.
27. A. Chambert-Loir and A. Thuillier, *Mesures de Mahler et équidistribution logarithmique*, *Ann. Inst. Fourier (Grenoble)* **59** (2009), no. 3, 977–1014.
28. Z. Chatzidakis and E. Hrushovski, *Difference fields and descent in algebraic dynamics*. I, *J. Inst. Math. Jussieu* **7** (2008), no. 4, 653–686; II, 687–704.
29. D. A. Cox, J. Little, and D. O’Shea, *Using algebraic geometry*, 2nd ed., *Grad. Texts in Math.*, vol. 185, Springer, New York, 2005.
30. L. DeMarco, *Dynamics of rational maps: Lyapunov exponents, bifurcations, and capacity*, *Math. Ann.* **326** (2003), no. 1, 43–73.
31. L. Denis, *Points périodiques des automorphismes affines*, *J. Reine Angew. Math.* **467** (1995), 157–167.
32. I. Dolgachev, *Lectures on invariant theory*, *London Math. Soc. Lecture Note Ser.*, vol. 296, Cambridge Univ. Press, Cambridge, 2003.
33. A. Douady and J. H. Hubbard, *Exploring the Mandelbrot set. The Orsay notes*, available at www.math.cornell.edu/~hubbard/OrsayEnglish.pdf.
34. ———, *Itération des polynômes quadratiques complexes*, *C. R. Acad. Sci. Paris Sér. I Math.* **294** (1982), no. 3, 123–126.
35. ———, *Étude dynamique des polynômes complexes. Partie II*, with the collaboration of P. Lavaurs, Tan Lei, and P. Sentenac, *Publ. Math. Orsay*, vol. 85, Univ. Paris-Sud, Dép. de Mathématiques, Orsay, 1985.
36. ———, *A proof of Thurston’s topological characterization of rational functions*, *Acta Math.* **171** (1993), no. 2, 263–297.
37. A. Epstein, *Integrality and rigidity for postcritically finite polynomials*, available at [arXiv:1010.2780](https://arxiv.org/abs/1010.2780).
38. G. Everest and T. Ward, *Heights of polynomials and entropy in algebraic dynamics*, *Universitext*, Springer, London, 1999.
39. N. Fakhruddin, *Boundedness results for periodic points on algebraic varieties*, *Proc. Indian Acad. Sci. Math. Sci.* **111** (2001), no. 2, 173–178.
40. ———, *Questions on self maps of algebraic varieties*, *J. Ramanujan Math. Soc.* **18** (2003), no. 2, 109–122.
41. G. Faltings, *Finiteness theorems for abelian varieties over number fields*, *Arithmetic Geometry* (Storrs, CT, 1984), Springer, New York, 1986, pp. 9–27. Translated from the German original [*Invent. Math.* **73** (1983), no. 3, 349–366; Erratum, **75** (1984), no. 2, 381] by E. Shipz.
42. E. V. Flynn, B. Poonen, and E. F. Schaefer, *Cycles of quadratic polynomials and rational points on a genus-2 curve*, *Duke Math. J.* **90** (1997), no. 3, 435–463.
43. D. Ghioca, L.-C. Hsia, and T. J. Tucker, *Preperiodic points for families of polynomials*, available at [arXiv:1102.2769](https://arxiv.org/abs/1102.2769).
44. D. Ghioca and Michael. E. Zieve, *Lattès maps in arbitrary characteristic*, in preparation.
45. W. Green, *Heights in families of abelian varieties*, *Duke Math. J.* **58** (1989), no. 3, 617–632.
46. P. Habegger, *Intersecting subvarieties of abelian varieties with algebraic subgroups of complementary dimension*, *Invent. Math.* **176** (2009), no. 2, 405–447.
47. ———, *On the bounded height conjecture*, *Int. Math. Res. Not. IMRN* **5** (2009), 860–886.
48. M. Hindry and J. H. Silverman, *Diophantine geometry: An introduction*, *Grad. Texts in Math.*, vol. 201, Springer, New York, 2000.
49. B. Hutz, *Good reduction of periodic points on projective varieties*, *Illinois J. Math.* **53** (2009), no. 4, 1109–1126.
50. ———, *Dynatomic cycles for morphisms of projective varieties*, *New York J. Math.* **16** (2010), 125–159.
51. ———, *Effectivity of dynatomic cycles for morphisms of projective varieties using deformation theory*, *Proc. Amer. Math. Soc.*, to appear, available at [arXiv:1011.5155](https://arxiv.org/abs/1011.5155).
52. D. Huybrechts, *Compact hyperkähler manifolds*, *Calabi–Yau Manifolds and Related Geometries* (Nordfjordeid, 2001), *Universitext*, Springer, Berlin, 2003, pp. 161–225.
53. P. Ingram, *A finiteness result for post-critically finite polynomials*, *Int. Math. Res. Not. IMRN*, to appear, available at [arXiv:1010.3393](https://arxiv.org/abs/1010.3393).
54. ———, *Variation of the canonical height for a family of polynomials*, available at [arXiv:1003.4225](https://arxiv.org/abs/1003.4225).
55. ———, *Canonical heights for Hénon maps*, preprint.

56. J.-P. Jouanolou, *Le formalisme du résultant*, Adv. Math. **90** (1991), no. 2, 117–263.
57. S. Kawaguchi, *Canonical height functions for affine plane automorphisms*, Math. Ann. **335** (2006), no. 2, 285–310.
58. ———, *Local and global canonical height functions for affine space regular automorphisms*, available at [arXiv:0909.3573](https://arxiv.org/abs/0909.3573).
59. S. Kawaguchi and J. H. Silverman, *Canonical heights and the arithmetic complexity of morphisms on projective space*, Pure Appl. Math. Q. **5** (2009), no. 4, 1201–1217. Special issue in honor of John Tate.
60. ———, *Nonarchimedean Green functions and dynamics on projective space*, Math. Z. **262** (2009), no. 1, 173–197.
61. A. Khrennikov, *Non-Archimedean analysis: quantum paradoxes, dynamical systems and biological models*, Math. Appl., vol. 427, Kluwer, Dordrecht, 1997.
62. S. Lang, *Fundamentals of Diophantine geometry*, Springer, New York, 1983.
63. E. Lau and D. Schleicher, *Internal addresses in the Mandelbrot set and irreducibility of polynomials*, Technical Report ims94-19, Stony Brook Institute for Mathematical Sciences, 1994.
64. C. Lee, *An upper bound for the height for regular affine automorphisms of \mathbb{A}^n* , available at [arXiv:0909.3107](https://arxiv.org/abs/0909.3107).
65. ———, *Height estimates for rational maps*, ProQuest LLC, Ann Arbor, MI, 2010. Ph.D. Thesis, Brown University.
66. A. Levy, *The space of morphisms on projective space*, Acta Arith. **146** (2011), no. 1, 13–31.
67. R. Mañé, P. Sad, and D. Sullivan, *On the dynamics of rational maps*, Ann. Sci. École Norm. Sup. (4) **16** (1983), no. 2, 193–217.
68. M. Manes, *\mathbb{Q} -rational cycles for degree-2 rational maps having an automorphism*, Proc. Lond. Math. Soc. (3) **96** (2008), no. 3, 669–696.
69. S. Marcello, *Sur la dynamique arithmétique des automorphismes affines*, Ph.D. thesis, Université Paris 7, 2000.
70. ———, *Sur les propriétés arithmétiques des itérés d'automorphismes réguliers*, C. R. Acad. Sci. Paris Sér. I Math. **331** (2000), no. 1, 11–16.
71. D. Masser and U. Zannier, *Torsion anomalous points and families of elliptic curves*, C. R. Math. Acad. Sci. Paris **346** (2008), no. 9-10, 491–494.
72. B. Mazur, *Modular curves and the Eisenstein ideal*, Inst. Hautes Études Sci. Publ. Math. **47** (1977), 33–186 (1978).
73. C. T. McMullen, *Families of rational maps and iterative root-finding algorithms*, Ann. of Math. (2) **125** (1987), no. 3, 467–493.
74. ———, *Complex dynamics and renormalization*, Ann. of Math. Stud., vol. 135, Princeton Univ. Press, Princeton, NJ, 1994.
75. ———, *Dynamics on K3 surfaces: Salem numbers and Siegel disks*, J. Reine Angew. Math. **545** (2002), 201–233.
76. C. T. McMullen and D. P. Sullivan, *Quasiconformal homeomorphisms and dynamics. III: The Teichmüller space of a holomorphic dynamical system*, Adv. Math. **135** (1998), no. 2, 351–395.
77. L. Merel, *Bornes pour la torsion des courbes elliptiques sur les corps de nombres*, Invent. Math. **124** (1996), no. 1-3, 437–449.
78. J. Milnor, *Geometry and dynamics of quadratic rational maps*, with an appendix by J. Milnor and L. Tan, Experiment. Math. **2** (1993), no. 1, 37–83.
79. ———, *On rational maps with two critical points*, Experiment. Math. **9** (2000), no. 4, 481–522.
80. ———, *On Lattès maps*, Dynamics on the Riemann Sphere (Holbæk, 2003) (P. G. Hjorth and C. L. Petersen, eds.), Eur. Math. Soc., Zürich, 2006, pp. 9–43.
81. A. Mimar, *On the preperiodic points of an endomorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ which lie on a curve*, Ph.D. Thesis, Columbia University, 1997.
82. P. Morton, *Arithmetic properties of periodic points of quadratic maps*, Acta Arith. **62** (1992), no. 4, 343–372.
83. ———, *On certain algebraic curves related to polynomial maps*, Compositio Math. **103** (1996), no. 3, 319–350.
84. P. Morton and J. H. Silverman, *Rational periodic points of rational functions*, Internat. Math. Res. Notices **2** (1994), 97–110.

85. D. Mumford, *On the equations defining abelian varieties. I*, Invent. Math. **1** (1966), 287–354.
86. ———, *On the equations defining abelian varieties. II*, Invent. Math. **3** (1967), 75–135.
87. ———, *On the equations defining abelian varieties. III*, Invent. Math. **3** (1967), 215–244.
88. D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, 3rd ed., Ergeb. Math. Grenzgeb. (2), vol. 34, Springer, Berlin, 1994.
89. D. Mumford and K. Suominen, *Introduction to the theory of moduli*, Algebraic Geometry, Oslo, 1970 (Oslo, 1970) (F. Oort, ed.), Wolters-Noordhoff, Groningen, 1972, pp. 171–222.
90. W. Narkiewicz, *Polynomial mappings*, Lecture Notes in Math., vol. 1600, Springer, Berlin, 1995.
91. D. G. Northcott, *Periodic points on an algebraic variety*, Ann. of Math. (2) **51** (1950), 167–177.
92. Y. Okuyama, *Repelling periodic points and logarithmic equidistribution in non-archimedean dynamics*, available at [arXiv:1106.3363](https://arxiv.org/abs/1106.3363).
93. F. Oort, *Canonical liftings and dense sets of CM-points*, Arithmetic Geometry (Cortona, 1994) (F. Catanese, ed.), Sympos. Math., vol. 37, Cambridge Univ. Press, Cambridge, 1997, pp. 228–234.
94. F. Pakovich, *Conservative polynomials and yet another action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on plane trees*, J. Théor. Nombres Bordeaux **20** (2008), no. 1, 205–218.
95. C. Petsche, L. Szpiro, and M. Tepper, *Isotriviality is equivalent to potential good reduction for endomorphisms of \mathbb{P}^N over function fields*, J. Algebra **322** (2009), no. 9, 3345–3365.
96. C. Petsche, L. Szpiro, and T. J. Tucker, *A dynamical pairing between two rational maps*, Trans. Amer. Math. Soc., to appear, available at [arXiv:0911.1875](https://arxiv.org/abs/0911.1875).
97. J. Pila, *O-minimality and the André–Oort conjecture for \mathbb{C}^n* , Ann. of Math. (2) **173** (2011), no. 3, 1779–1840.
98. R. Pink, *A common generalization of the conjectures of André–Oort, Manin–Mumford, and Mordell–Lang* (2005), unpublished.
99. K. Schmidt, *Dynamical systems of algebraic origin*, Progr. Math., vol. 128, Birkhäuser, Basel, 1995.
100. J.-P. Serre, *Local fields*, Grad. Texts in Math., vol. 67, Springer, New York, 1979. Translated from the French by M. J. Greenberg.
101. J.-P. Serre and J. Tate, *Good reduction of abelian varieties*, Ann. of Math. (2) **88** (1968), 492–517.
102. N. Sibony, *Dynamique des applications rationnelles de \mathbf{P}^k* , Dynamique et géométrie complexes (Lyon, 1997), Panor. Synthèses, vol. 8, Soc. Math. France, Paris, 1999, pp. ix–x, xi–xii, 97–185.
103. J. H. Silverman, *Heights and the specialization map for families of abelian varieties*, J. Reine Angew. Math. **342** (1983), 197–211.
104. ———, *Computing heights on elliptic curves*, Math. Comp. **51** (1988), no. 183, 339–358.
105. ———, *Rational points on K3 surfaces: a new canonical height*, Invent. Math. **105** (1991), no. 2, 347–373.
106. ———, *Geometric and arithmetic properties of the Hénon map*, Math. Z. **215** (1994), no. 2, 237–250.
107. ———, *The field of definition for dynamical systems on \mathbf{P}^1* , Compositio Math. **98** (1995), no. 3, 269–304.
108. ———, *Computing canonical heights with little (or no) factorization*, Math. Comp. **66** (1997), no. 218, 787–805.
109. ———, *The space of rational maps on \mathbf{P}^1* , Duke Math. J. **94** (1998), no. 1, 41–77.
110. ———, *The arithmetic of dynamical systems*, Grad. Texts in Math., vol. 241, Springer, New York, 2007.
111. ———, *The arithmetic of elliptic curves*, 2nd ed., Grad. Texts in Math., vol. 106, Springer, Dordrecht, 2009.
112. ———, *Height estimates for equidimensional dominant rational maps*, J. Ramanujan Math. Soc. **26** (2011), no. 2, 145–163.
113. ———, *An algebraic approach to certain cases of Thurston rigidity*, Proc. Amer. Math. Soc., to appear, available at [arXiv:1010.4562](https://arxiv.org/abs/1010.4562).
114. D. Speyer, *Rational maps with all critical points fixed*, MathOverflow, mathoverflow.net/questions/4983.

115. M. Stoll, *Rational 6-cycles under iteration of quadratic polynomials*, LMS J. Comput. Math. **11** (2008), 367–380.
116. L. Szpiro and T. J. Tucker, *A Shafarevich–Faltings theorem for rational functions*, Pure Appl. Math. Q. **4** (2008), no. 3, 715–728.
117. J. Tate, *Variation of the canonical height of a point depending on a parameter*, Amer. J. Math. **105** (1983), no. 1, 287–294.
118. D. Tischler, *Critical points and values of complex polynomials*, J. Complexity **5** (1989), no. 4, 438–456.
119. C. Voisin, *Géométrie des espaces de modules de courbes et de surfaces K3 (d’après Gritsenko-Hulek-Sankaran, Farkas-Popa, Mukai, Verra, et al.)*, Astérisque **317** (2008), x, 467–490. Séminaire Bourbaki. Vol. 2006/2007, Exp. No. 981.
120. M. Waldschmidt, *Algebraic dynamics and transcendental numbers*, Noise, Oscillators and Algebraic Randomness (Chapelle des Bois, 1999) (M. Planat, ed.), Lecture Notes in Phys., vol. 550, Springer, Berlin, 2000, pp. 372–378.
121. L. Wang, *Rational points and canonical heights on K3-surfaces in $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$* , Recent developments in the inverse Galois problem (Seattle, WA, 1993) (M. D. Fried, S. S. Abhyankar, W. Feit, Y. Ihara, and H. Völklein, eds.), Contemp. Math., vol. 186, Amer. Math. Soc., Providence, RI, 1995, pp. 273–289.
122. S. Zhang, *Small points and adelic metrics*, J. Algebraic Geom. **4** (1995), no. 2, 281–300.
123. M. E. Zieve, *Cycles of polynomial mappings*, ProQuest LLC, Ann Arbor, MI, 1996. Ph.D. Thesis, University of California, Berkeley.
124. B. Zilber, *Exponential sums equations and the Schanuel conjecture*, J. London Math. Soc. (2) **65** (2002), no. 1, 27–44.
125. K. Zsigmondy, *Zur Theorie der Potenzreste*, Monatsh. Math. Phys. **3** (1892), no. 1, 265–284.

Additional References

A reasonably up-to-date list of references for arithmetic dynamics is maintained at

<http://math.brown.edu/~jhs/ADSHome.html>

Index

- abelian subgroup, bounds order of size of
 - group, 35
- abelian variety
 - complex multiplication, 88, 110
 - moduli space of principally polarized, 88, 110
 - Néron–Ogg–Shafarevich criterion, 47
 - torsion point, 88
 - uniform boundedness, 10
- absolute height, 69
- absolute value, 69
 - function field, 45
- action, closed, 14
- affine automorphism, 40
 - locus of indeterminacy, 40
 - regular, 40, 82, 83
 - regular is algebraically stable, 40
- affine cone, 16
- affine map, 17
- affine morphism, 40
- affine space, quotient by symmetric group, 11
- algebraic group, 12
 - geometrically reductive, 14
 - linear, 14
 - linearly reductive, 14
 - 1-parameter subgroup, 17
 - radical of, 14
 - reductive, 14
 - semistable point, 16
 - stable point, 16
 - unstable point, 16
- algebraic number, 88
- algebraic stability, 40
- algebraic variety, equidistributed set of points, 99
- alternating group, 36
- ample cone, 87
- André–Oort conjecture, 110, 111
- Arakelov theory, 73
- Arakelov–Zhang pairing, 73
- automorphism
 - affine, 40
 - birational map between K3 surfaces is, 52
 - of infinite order, 50
 - regular affine, 40, 82, 83
 - regular affine is algebraically stable, 40
 - trivial for most maps, 36
 - automorphism group, 34, 65, 114
 - abelian subgroup of, 35
 - bound for size of, 35, 36
 - cyclic, 36
 - is finite, 34
 - locus in M_2 with nontrivial, 40, 65
 - of degree two map, 65
 - of projective space, 8
 - trivial if no twists, 114
- Baker, M., 46, 73, 80, 111
- Baragar, A., 88
- Becker, P.-G., 88
- Benedetto, R., 46, 73
- Bergweiler, W., 88
- Berkovich space, 77, 109
 - invariant measure on, 108
- Bertrand, D., 88
- Bhatnagar, A., 75, 121
- BiCrit_d, 43
 - is isomorphic to \mathbb{A}^2 , 43
- birational map, between K3 surfaces is
 - automorphism, 52
- birationally isotrivial family, 73
- Birch–Swinnerton-Dyer conjecture, 65
- Bombieri, E., 80
- Böttcher theorem, 88
- bounded height, 69, 84–86
- Bousch, T., 60
- Brauer group, 118
- Brolin–Lyubich measure, 108
- Buff, X., 60, 98, 99
- Call, G., 79
- canonical bundle, 52
- canonical height, 69, 70
 - computing numerically, 77
 - critical, 100
 - for regular affine automorphism, 83
 - is algebraic number, 89

- is transcendental?, 88, 89
- Lehmer conjecture, 101
- local, *see* local canonical height
- local decomposition, 76
- on K3 surface, 84, 86
- over function field, 73, 78
- p -adic, 88
- pairing, 72
- periodic point has zero, 86
- preperiodic point has zero, 72, 75
- properties of, 70
- varying in a family, 77, 78
- zero over function field, 75
- categorical quotient, 13
 - by reductive group exists, 14, 17
 - is quasi-projective, 17
 - properties, 13, 24
- Cauchy residue theorem, 30
- Cauchy sequence, 71
- Chambert-Loir, A., 73
- Chatzidakis, Z., 73
- Chebyshev polynomial, 88, 91
- chordal derivative, 108
- chordal metric, 108
 - p -adic, 109
- closed action, 14
- closed orbit, 14
- closure of orbit, 16
- CM, 88, 110
- coarse moduli space, 5, 60
- cocycle, 115
 - associated to field of moduli, 117
 - associated to twist, 115, 116
 - cohomologous, 115
- cohomologous cocycle, 115
- cohomology group, 115
 - of μ_n , 115
- cohomology set, 115
- complex multiplication, 88, 110
- cone, affine, 16
- conjugate maps, 4, 20
- conjugation
 - commutes with iteration, 4, 8
 - multiplier is invariant, 25
 - transforms orbit, 4
- connectivity, categorical quotient preserves, 13
- conservative map, 109
- conservative polynomial
 - number of, 110
- constant moduli, 45
- constructible descent, 74
- critical canonical height, 100
 - commensurable with moduli height, 101, 104
 - Lehmer conjecture, 101
- critical point, 91
 - all fixed, 109
 - is algebraic number, 88
 - maps with exactly two, 43
 - moduli space with marked, 92, 111
 - number of, 91
 - relation to Lyapunov exponent, 108
 - sum of canonical heights, 100, 101
- critical point portrait, 92
 - conditions are transversal, 93
 - finitely many maps with given, 92, 109
 - partial, 99
- cyclic group, 36
- cyclotomic polynomial, 26, 57, 63
- Dedekind domain, 49
- degree of a rational map, 40
- DeMarco, L., 80, 111, 112, 121
- Denis, L., 84
- derivative, 91
 - chordal, 108
- dihedral group, 36
- discrete dynamical system, 3
- discrete valuation ring, 47
- divisor
 - ample, 85, 86
 - height associated to, 85
- dominant rational map, 7, 81
 - height bound, 81
 - Zariski open subset of Hom_d^N , 82
- Douady, A., 60, 88
- DVR, *see* discrete valuation ring
- dynamical André–Oort conjecture, 110, 111
- dynamical canonical height, *see* canonical height
- dynamical modular curve, 59, 60, 65
 - is fine moduli space for $n \geq 2$, 60
 - is moduli space, 59
 - nonsingular, 61
 - not fine moduli space, 67
 - of genus zero, 59
 - rational points on, 65
 - reducible, 66
- dynamotic polynomial, 26, 57
 - factored using cyclotomic polynomials, 63
 - for $z^2 + c$, 59
 - for $z^2 - \frac{3}{4}$, 58
 - higher dimensional analogue, 58
 - is a polynomial, 58
 - of a polynomial, 57
- dynamotic zero cycle, 58
 - is effective, 58
- effective zero cycle, 58
- elementary symmetric polynomial, 11, 104
- elliptic curve, 5
 - good reduction, 49
 - isomorphism class of, 113
 - moduli space, 5
 - Mordell–Weil group, 113

- uniform boundedness, 9
- elliptic modular curve, 65
- Epstein, A., 96–99, 121
- equidistribution, Zariski, 99
- equilibrium measure, 108
- equivalent maps, 4, 20
- everywhere good reduction, 45, 49
 - potential, 45
- exact period, 4
 - relation to formal period, 58
- Faber, X., 101
- Fakhruddin, N., 10
- Faltings, G., 49
- Fatou set, 89
- field of definition, 113, 116
 - contains field of moduli, 116
 - field of moduli is for even degree maps, 118
 - field of moduli is for polynomials, 118
 - when is field of moduli a, 117
- field of genus one, 54
- field of moduli, 113, 116
 - cocycle associated to, 117
 - contained in field of definition, 116
 - is field of definition for even degree maps, 118
 - is field of definition for polynomials, 118
 - when is it a field of definition, 117
- filled Julia set, 76
- fine moduli space, 6, 34, 60
 - modular curve not, 67
- finite group, acting on a variety, 11
- fixed point, 4, 36
 - critical point are all, 109
 - totally ramified, 42
- flat morphism, 102
- flexible Lattès map, 53
- Flynn, E. V., 9, 65
- FOD, *see* field of definition
- FOM, *see* field of moduli
- formal period, 4, 8, 58, 66, 93
 - relation to exact period, 58
- forward orbit, 3
- function field, 45
 - absolute values on, 45
 - canonical height over, 73, 78
 - canonical height zero, 75
 - points of bounded height, 70, 85
- function, rational, 6
- Galois group, acts on rational map, 114
- general linear group
 - projective, 8
 - subgroup of, 14
- genus
 - dynamical modular curve of genus zero, 59
 - field of genus one, 54
 - Hurwitz formula, 43, 91
 - of $X_1(n)$, 61
- geometric invariant theory, 12
 - fundamental theorem of, 17
- geometric quotient, 13
 - by reductive group exists, 14, 17
 - properties, 24
- geometrically reductive algebraic group, 14
- Ghioca, D., 53, 56, 80, 101, 112
- G -invariant function, 14, 16
- GIT, *see* geometric invariant theory
- Gleason, A., 95
- G -linearization, 15
- good reduction, 45, 47, 76
 - everywhere, 45, 49
 - everywhere potential, 45
 - iff $\varepsilon_v(\phi) = 0$, 48
 - of elliptic curve, 49
 - potential, 45, 47, 49
- Green function, 76, 101, 105, 108
- group
 - algebraic, 12
 - alternating, 36
 - cyclic, 36
 - dihedral, 36
 - finite, acting on a variety, 11
 - linear algebraic, 14
 - order bounded in terms of abelian subgroups, 35
 - symmetric, 36
 - wreath product, 61
- group cohomology, 115
- group of self-similarities, 34
- Habegger, P., 80
- harmonic function, 76
- Hausdorff topology, 41
- Hecke correspondence, 111
- height
 - additivity, 85
 - associated to a divisor, 85
 - bound for dominant rational map, 81
 - canonical, *see* canonical height
 - expansion coefficient, 82
 - finitely many points of bounded, 69, 84–86
 - for $k(T)$, 70
 - functoriality, 85
 - local, 76, 101, 105, 108
 - of a polynomial, 110
 - of a rational map, 70, 103
 - preperiodic point has bounded, 71
 - pullback by flat morphism, 102
 - relation for regular affine automorphism, 83
 - transformation by morphism, 70
 - varying in a family, 77, 78
 - vector-valued, 88

- Weil, 69
- Weil height machine, 85
- height expansion coefficient, 82
 - of regular affine automorphism, 82
 - uniformly positive, 82
- height machine, 85
- Hénon map, 40, 75, 80, 83, 84
- Hilbert polynomial, 99
- Hilbert scheme, 75
- Hilbert “theorem 90”, 114, 115, 118
- Hilbert theorem on finitely generated algebras, 11, 12
- Hölder continuous, 77
- Hom_d , 3, 6
 - dimension of, 21
 - is subvariety of \mathbb{P}^{2d+1} , 7
 - semistable locus, 21
 - stable locus, 21
- Hom_d^n , 7
 - action of SL_{n+1} , 20
 - complement is irreducible hypersurface, 7
 - dimension of, 24
 - dominant maps Zariski open, 82
 - good reduction of map in, 45, 47
 - is subvariety of \mathbb{P}^N , 7
 - isotrivial family in, 45
 - trivial family in, 45
- homogeneous space, 118
- Hrushovski, E., 73
- Hsia, L.-C., 80, 112
- Hubbard, J., 60, 88
- Hurwitz genus formula, 43, 91
- Hutz, B., 58
- hyperbolic component, 61
- ideal sheaf, 99
- image of rational map, 81
- imprimitive dynamical system, 74
- indeterminacy locus, 40, 78, 81
- inertia group, 47
- Ingram, P., 75, 79, 80, 84, 104, 111
- invariant function, 14, 16
- invariant measure, 107, 108
 - p -adic, 108
- invertible sheaf, linearization of, 15
- involution, 50
 - on K3 surface, 51, 84
- irreducibility, quotient preserves, 13
- isotrivial family, 45, 73
- itinerary map, 61
- Jacobian variety, 65
- j -invariant, 5
- J -stability, 41, 42
- Julia set, 61, 76, 101
 - periodic points dense in, 107
- K3 surface, 50, 52, 84
 - ample cone, 87
 - birational map is automorphism, 52
 - canonical height, 86
 - contained in $(\mathbb{P}^1)^3$, 53
 - contained in $(\mathbb{P}^2)^2$, 50
 - finitely many periodic points over
 - number field, 87
 - involutions on, 51, 84
 - moduli space, 52
 - Picard group, 84
 - uniform boundedness conjecture, 87
- Kawaguchi, S., 72, 83
- K -equivalence, 113
- K -twist, *see* twist
- Kummer sequence, 115, 118
- Lang, S., 105
- Lattès map, 9, 27, 53, 89, 91–93, 101
 - flexible, 53
 - Milnor multiplier function, 27
 - multipliers of, 27
 - realization of, 53, 54
 - reduced realization, 54
 - rigid, 53
- Lau, E., 60
- Laurent series, 62
- Lee, C.-G., 82, 83, 121
- Lefschetz principle, 97
- Lehmer conjecture, 101
- Lei, T., 60
- level structure, 6, 8, 66
- Levy, A., 23, 28, 29, 35, 36, 67, 121
- limited subset, 74
- line bundle, linearization of, 15
- linear algebraic group, 14
 - geometrically reductive, 14
 - linearly reductive, 14
 - reductive, 14
- linear equivalence, 85
- linearization, 15
- linearly reductive algebraic group, 14
- local canonical height, 76, 101, 105, 108
 - computing numerically, 77
 - decomposes canonical height, 76
 - for maps on \mathbb{P}^N , 76
 - is Hölder continuous, 77
 - on Berkovich space, 77
 - properties of, 76
- locus of indeterminacy, 40, 78
- logarithmic height, 69
- Lyapunov exponent, 101, 107, 108
 - conjugation invariant, 108
 - p -adic, 109
 - relation with critical points, 108
 - relation with periodic points, 109
- Lyapunov, A., 108
- Lyubich measure, 108
- Macaulay resultant, 7, 47
 - homogeneity of, 48

- Mandelbrot set, 61
 - is connected, 88
- Mañé, R., 98
- Manes, M., 61, 66, 67, 121
- map
 - periodic point, 4
 - preperiodic point, 4
 - semiconjugate, 5
- Marcello, S., 84
- Masser, D., 80
- Mazur, B., 9
- McMullen's theorem, 27, 53
 - in characteristic p , 28, 29
 - in higher dimension, 29
- McMullen, C., 27, 42
- measure of maximal entropy, 108
- measure, invariant, 108
- Merel, L., 9
- metric
 - chordal, 109
- metric, chordal, 108
- Miller, G. A., 35
- Milnor multiplier function, 26
 - determines map up to conjugacy, 27, 53
 - on moduli space M_d^n , 28
- Milnor, J., 30, 33, 43, 53
- Mimar, A., 73
- minimal model, 47
- minimal resultant, 47
 - exponent of, 48
 - exponent zero iff good reduction, 48
- Möbius function, 26, 57
- model theory, 111
- modular curve
 - dynamical, 59, 60
 - elliptic, 65
 - level structure, 6, 8, 66
 - reducible, 66
 - $X_0(n)$, 60, 61
 - $X_1(n)$, 59, 61
 - $Y_0(n)$, 60
 - $Y_1(n)$, 59, 61
- moduli problem, 5
- moduli space, 5
 - coarse, 5, 60
 - construction of, 20
 - dynamical modular curve is, 59
 - elliptic curve, 5
 - fine, 6, 34, 60
 - for maps of \mathbb{P}^n , 8
 - of K3 surfaces, 52
- moduli space A_g , 88, 110
- moduli space M_d
 - critical canonical height is function on, 100
 - dimension of, 21
 - exists as geometric quotient, 21
 - finite map to affine space via multipliers, 27
 - height of points on, 101, 104
 - is rational, 36
 - locus of maps with nontrivial Aut , 40
 - locus of maps with nontrivial automorphism group, 65
 - locus of polynomial maps in M_2 , 39
 - locus of polynomial maps in M_d , 42, 96
 - $M_2 \cong \mathbb{A}^2$, 30
 - \overline{M}_2 , 33
 - Milnor multiplier function on, 26
 - PCF maps are Zariski dense, 98
 - PCF maps are Zariski equidistributed?, 99
 - PCF maps have bounded height, 101
 - Schwarzian derivative as function on, 29
 - semistable locus, 21
 - stable locus, 21
 - structure sheaf, 21
- moduli space M_d , 8
- moduli space M_d^{crit} , 92, 111
- moduli space M_d^n
 - dimension of, 24
 - exists as geometric quotient, 23
 - is not fine, 34
 - is rational?, 36
 - Milnor multiplier function on, 28
 - scheme over \mathbb{Z} , 47
- moduli space M_d^n , 8
- Montel's theorem, 98
- Mordell–Weil group, 113
- morphism
 - conjugate, 4, 20
 - equivalent, 4, 20
 - flat, 102
- Morton, P., 9, 60, 65
- $M_{\mathbb{Q}}$ -bounded function, 105
- $M_{\mathbb{Q}}$ -constant, 105
- $\mu^{\mathcal{L}}$, 18
- multiplicative group, 15, 17
- multiplier, 24
 - as map on tangent space, 25, 28, 91
 - equals one, 25
 - invariant under conjugation, 25
 - Milnor function, 26
 - of Lattès map, 27
 - symmetric function of, 26
- multiplier spectrum, 25
 - determine map up to conjugacy, 27, 53
 - $1 \notin \Lambda_{\phi}^n \Rightarrow$ periodic points distinct, 25
- Mumford, D., 10, 12
- Mumford's numerical criterion, 18, 22
- Néron–Ogg–Shafarevich criterion, 47
- Noether, E., 39
- normal forms lemma, 30
 - as multipliers go to infinity, 34

- Northcott, D. G., 71
 Nullstellensatz, 70, 106
 numerical criterion for (semi)stability, 18,
 22
 diagonalized form, 19
- \mathcal{O} -minimal geometry, 111
- 1-cocycle, 115
 associated to field of moduli, 117
 associated to twist, 115, 116
- 1-parameter subgroup, 17
 specialization, 17
- orbit, 3, 12
 closed, 14
 closure of, 16
 forward, 3
 scheme theoretic, 12
 transformed by conjugation, 4
 under semigroup generated by two maps,
 80
 Zariski closure, 13
- Pakovich, F., 110
- parameter space, elliptic curve, 5
- PCF map, *see* postcritically finite map
- period
 exact, 4
 formal, 4, 8
- periodic point, 4
 condition for distinct, 25
 dense in Julia set, 107
 dense in \mathbb{P}^n , 35
 exact period, 4
 exact period versus formal period, 58
 formal period, 4, 58, 66, 93
 has bounded height, 71
 has canonical height zero, 72, 86
 is algebraic number, 88
 multiplier, 24
 number of nonrepelling, 107
 on K3 surface, 86
 relation to Lyapunov exponent, 109
 uniform boundedness conjecture, 9, 84,
 87
- Petsche, C., 23, 34, 45, 72
- PGL_{n+1} , 8
 subgroups of PGL_2 , 36
- Picard group, 84
- Pila, J., 111
- Pink, R., 80
- polynomial
 bihomogeneous, 50
 Chebyshev, 88
 dynatomic, 57
 height of, 110
 normalized, 110
- polynomial map
 FOM = FOD, 118
 has totally ramified fixed point, 42
 locus in \mathcal{M}_2 , 39
 locus in \mathcal{M}_d , 42, 96
 with marked critical points, 96
- Poonen, B., 9, 29, 65
- portrait, 92
 partial, 99
- postcritical stability, 41, 42
- postcritically finite map, 91
 analogous to CM abelian variety, 110
 critical point portrait, 92, 99
 has bounded height, 101
 is algebraic point in \mathcal{M}_d , 88, 92
 Lehmer conjecture, 101
 p -adically integral in \mathcal{M}_d , 96
 Zariski dense in \mathcal{M}_d , 98
 Zariski equidistributed in \mathcal{M}_d ?, 99
- potential good reduction, 45, 47, 49
- power map, 26, 43, 88, 89, 91
- preperiodic point, 4
 dense in \mathbb{P}^n , 35
 formal period and tail length, 93
 has bounded height, 71
 has canonical height zero, 72, 75
 is algebraic number, 88
 specializations have bounded height, 78
 uniform boundedness conjecture, 9
- primitive dynamical system, 74
- projective line, moduli space of maps, 8
- projective linear group, 8
 subgroups of, 36
- projective space
 automorphism group, 8
 chordal metric, 108, 109
 covering by affine space, 16
 moduli space of maps, 8
- qc-stability, 41, 42
- quadratic polynomial
 Julia set, 61
 uniform boundedness conjecture, 9
- quotient
 categorical, 13
 geometric, 13
- radical, 14
- ramification index, 43, 91
- Rat_d^n , 7
 is subvariety of \mathbb{P}^N , 7
- rational function, 6
- rational map, 7
 action of Galois group on, 114
 automorphism group, 34, 65, 114
 canonical height is transcendental?, 89
 chordal derivative, 108
 conjugate of, 4, 8
 conservative, 109
 constant moduli, 45
 critical canonical height, 100

- critical point, 91
- degree of, 40
- derivative, 91
- dominant, 7, 81
- Fatou set, 89
- field of definition, 116
- field of moduli, 116
- fixed point, 36
- good reduction, 45, 47, 76
- good reduction iff $\varepsilon_v(\phi) = 0$, 48
- height expansion coefficient, 82
- height of, 70, 103
- induced map on tangent space, 28, 91
- invariant measure, 107, 108
- isotrivial family of, 45
- iteration of nonmorphism, 81
- Julia set, 61, 76, 101, 107
- K -equivalence, 113
- Lyapunov exponent, 101, 107, 108
- minimal model, 47
- minimal resultant, 47–49
- multiplier of periodic point, 24
- multiplier spectrum, 25
- nonisotrivial has nondegenerate canonical height, 73
- normal forms lemma for degree two, 30
- postcritically finite, 91
- ramification index of, 43, 91
- semigroup generated by two, 80
- trivial family of, 45
- twist, 49, 113
- with exactly two critical points, 43
- with nontrivial Aut group, 40
- rationally primitive dynamical system, 74
- reduced Lattès realization, 54
- reduced, categorical quotient preserves, 13
- reductive algebraic group, 14
 - 1-parameter subgroup, 17
 - semistable point, 16
 - stable point, 16
 - unstable point, 16
- regular affine automorphism, 40, 83
 - canonical height, 83
 - height expansion coefficient, 82
 - height relation, 83
 - Hénon map, 40, 83
 - is algebraically stable, 40
 - uniform boundedness conjecture, 84
- regular surface, 52
- residue theorem, 30
- resultant, 30, 63, 66, 109
 - Macaulay, 7, 47
 - minimal, 48, 49
 - Sylvester matrix, 95, 97
- Riemann–Roch theorem, 118
- rigid Lattès map, 53
- rigidity theorem, 92, 93, 109
 - algebraic proof, 95
 - for polynomials, 96
- ring of invariants, 11
- Rivera-Letelier, J., 107
- Sad, P., 98
- Salem number, 89
- Schaefer, E., 9, 65
- Schleicher, D., 60
- Schwarzian derivative, 29
- semiconjugate maps, 5
- semistable point, 16
 - categorical quotient of, 17
 - numerical criterion, 18
- Shafarevich, I., 49
- Silverman, J. H., 9, 30, 33, 72, 77, 79, 83, 96, 97, 107, 118
- special linear group
 - action on Hom_d^n , 20
- specialization, 17, 77, 78
 - preperiodic specializations have bounded height, 78
- Speyer, D., 109
- stability
 - J , 41, 42
 - postcritical, 41, 42
 - qc, 41, 42
 - topological, 41, 42
- stabilizer, 13
- stabilizer group, 34, 114
- stable point, 16
 - geometric quotient of, 17
 - numerical criterion, 18, 22
- Stoll, M., 9, 65
- strange attractor, 40
- structure sheaf, 14, 21
- Sullivan, D., 42, 98
- support, 98
- surface
 - $K3$, 50, 52
 - regular, 52
- Sylvester matrix, 95, 97
- symbolic dynamics, 61
- symmetric group, 11, 36
- symmetric polynomial, 11, 104
- Szpiro, L., 23, 34, 45, 72, 75
- tangent space, 25, 28, 91
- Tate, J., 70, 77, 79, 86
- Taylor series, 25, 58
- telescoping sum, 71, 86
- Tepper, M., 23, 34, 45, 121
- Thuillier, A., 73
- Thurston rigidity theorem, 92, 93, 109
 - algebraic proof, 95
 - for polynomials, 96
- Thurston transversality theorem, 93, 99
 - algebraic proof, 95
 - for polynomials, 96
 - in characteristic p , 97

- Thurston, W., 92, 93
 Tischler, D., 110
 topological stability, 41, 42
 topology, Hausdorff, 41
 transcendence theory, 111
 transcendental number, 88, 89
 transversality theorem, 93, 99
 - algebraic proof, 95
 - for polynomials, 96
 - in characteristic p , 97
 triangle inequality, 70, 106
 trivial family, 45
 Tucker, T., 72, 80, 101, 112
 twist, 49, 113
 - cocycle associated to, 115, 116
 - cohomological description of, 116
 - none if automorphism group is trivial, 114
 - of \mathbb{P}^1 , 118
 uniform boundedness
 - conjecture, 9, 84, 87
 - for K3 surfaces, 87
 - for preperiodic points, 9
 - for quadratic polynomials, 9
 - for regular affine automorphisms, 84
 - torsion on abelian variety, 10
 - torsion on elliptic curve, 9
 universal family, 6
 universally submersive map, 17
 unlikely intersection conjecture, 80
 unstable point, 16
 - numerical criterion, 18
 valuative criterion for properness, 33
 variety
 - action of algebraic group, 12
 - involution, 50
 - K3 surface, 50
 - quotient by finite group, 11
 - set of bounded height, 86
 vector space, stability of points, 16
 vector-valued height, 88

 Weil height, 69
 Weil height machine, 85
 wreath product, 61

 $X_0(n)$, 60
 - genus of, 64
 - is irreducible, 61 $X_1(n)$, 59
 - genus of, 61, 64
 - is irreducible, 61
 - rational points on, 65
 $Y_0(n)$, 60
 $Y_1(n)$, 59
 - is nonsingular, 61
 Zannier, U., 80
 Zariski equidistributed set of points, 99
 Zhang, S., 73
 Zieve, M., 53, 56, 58, 117
 Zilber, B., 80

Titles in This Series

- 30 **Joseph H. Silverman**, Moduli spaces and arithmetic dynamics, 2012
- 29 **Marcelo Aguiar and Swapneel Mahajan**, Monoidal functors, species and Hopf algebras, 2010
- 28 **Saugata Ghosh**, Skew-orthogonal polynomials and random matrix theory, 2009
- 27 **Jean Berstel, Aaron Lauve, Christophe Reutenauer, and Franco V. Saliola**, Combinatorics on words: Christoffel words and repetitions in words, 2009
- 26 **Victor Guillemin and Reyer Sjamaar**, Convexity properties of Hamiltonian group actions, 2005
- 25 **Andrew J. Majda, Rafail V. Abramov, and Marcus J. Grote**, Information theory and stochasticity for multiscale nonlinear systems, 2005
- 24 **Dana Schlomiuk, Andrei A. Bolibrukh, Sergei Yakovenko, Vadim Kaloshin, and Alexandru Buium**, On finiteness in differential equations and Diophantine geometry, 2005
- 23 **J. J. M. M. Rutten, Marta Kwiatkowska, Gethin Norman, and David Parker**, Mathematical techniques for analyzing concurrent and probabilistic systems, 2004
- 22 **Montserrat Alsina and Pilar Bayer**, Quaternion orders, quadratic forms, and Shimura curves, 2004
- 21 **Andrei Tyurin**, Quantization, classical and quantum field theory and theta functions, 2003
- 20 **Joel Feldman, Horst Knörrer, and Eugene Trubowitz**, Riemann surfaces of infinite genus, 2003
- 19 **L. Lafforgue**, Chirurgie des grassmanniennes, 2003
- 18 **G. Lusztig**, Hecke algebras with unequal parameters, 2003
- 17 **Michael Barr**, Acyclic models, 2002
- 16 **Joel Feldman, Horst Knörrer, and Eugene Trubowitz**, Fermionic functional integrals and the renormalization group, 2002
- 15 **José I. Burgos Gil**, The regulators of Beilinson and Borel, 2002
- 14 **Eyal Z. Goren**, Lectures on Hilbert modular varieties and modular forms, 2002
- 13 **Michael Baake and Robert V. Moody, Editors**, Directions in mathematical quasicrystals, 2000
- 12 **Masayoshi Miyanishi**, Open algebraic surfaces, 2001
- 11 **Spencer J. Bloch**, Higher regulators, algebraic K -theory, and zeta functions of elliptic curves, 2000
- 10 **James D. Lewis**, A survey of the Hodge conjecture, Second Edition, 1999
- 9 **Yves Meyer**, Wavelets, vibrations and scaling, 1998
- 8 **Ioannis Karatzas**, Lectures on the mathematics of finance, 1996
- 7 **John Milton**, Dynamics of small neural populations, 1996
- 6 **Eugene B. Dynkin**, An introduction to branching measure-valued processes, 1994
- 5 **Andrew Bruckner**, Differentiation of real functions, 1994
- 4 **David Ruelle**, Dynamical zeta functions for piecewise monotone maps of the interval, 1994
- 3 **V. Kumar Murty**, Introduction to Abelian varieties, 1993
- 2 **M. Ya. Antimirov, A. A. Kolyskin, and Rémi Vaillancourt**, Applied integral transforms, 1993
- 1 **D. V. Voiculescu, K. J. Dykema, and A. Nica**, Free random variables, 1992

This monograph studies moduli problems associated to algebraic dynamical systems. It is an expanded version of the notes for a series of lectures delivered at a workshop on *Moduli Spaces and the Arithmetic of Dynamical Systems* at the Bellairs Research Institute, Barbados, in 2010.

The author's goal is to provide an overview, with enough details and pointers to the existing literature, to give the reader an entry into this exciting area of current research. Topics covered include:

- (1) Construction and properties of dynamical moduli spaces for self-maps of projective space.
- (2) Dynamotic modular curves for the space of quadratic polynomials.
- (3) The theory of canonical heights associated to dynamical systems.
- (4) Special loci in dynamical moduli spaces, in particular the locus of post-critically finite maps.
- (5) Field of moduli and fields of definition for dynamical systems.



For additional information
and updates on this book, visit
www.ams.org/bookpages/crmm-30

AMS on the Web
www.ams.org