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Moduli Spaces and Arithmetic Dynamics

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Preface

This monograph is an expanded version of the notes for a series of lectures delivered at a workshop on *Moduli Spaces and the Arithmetic of Dynamical Systems*, Bellairs Research Institute, Barbados, May 2–9, 2010. As such, the level of exposition is uneven, with some results being worked out in detail, while others are merely sketched or have proofs by citation. The goal is to provide an overview, with enough details and pointers to the existing literature, to give the reader an entry into this exciting area of current research. It is the author's hope that this will be useful, especially since at present there are only a small number of books [4, 38, 61, 90, 99, 110] dealing with the arithmetic or algebraic side of dynamical systems. For further reading, the reader might consult the webpage

`http://www.math.brown.edu/~jhs/ADSHome.html`

which contains links to an extensive list of articles in this area.

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Joseph H. Silverman
August 2011

Schedule of Talks at the Bellairs Workshop

- **Monday:**
  - Joseph Silverman: *Introduction to Moduli Spaces*
  - Laura DeMarco: *Moduli Spaces of One-Dimensional Dynamical Systems*
- **Tuesday:**
  - Joseph Silverman: *Dynamical Moduli Spaces*
  - Adam Epstein: *Transversality and Holomorphic Dynamics*
- **Wednesday:**
  - Joseph Silverman: *Dynatomic Polynomials and Dynamical Modular Curves*
  - Michelle Manes: *Level Structures on Dynamical Moduli Spaces*
- **Thursday:**
  - Joseph Silverman: *Canonical Heights*
    - Anupam Bhatnagar: *Points of Canonical Height Zero on Projective Varieties*
    - ChongGyu (Joey) Lee: *Height Estimates for Dominant Morphisms*
    - Alon Levy: *Semistable Reduction for Dynamical Systems*
- **Friday:**
  - Joseph Silverman: *Field of Moduli and Fields of Definition*
  - Michael Tepper: *Isotriviality and* M_d^n

Glossary

$[F_n, G_n]$	nth iterate of $[F, G]$, 57
$[\phi, \psi]_{\mathrm{AZ}}$	Arakelov–Zhang pairing, 72
$[\phi, \psi]_{\mathrm{KS}}$	Kawaguchi–Silverman pairing, 72
$F \asymp G$	notation for $F \ll G$ and $G \ll F$, 101
$g \cdot x$	the image of the action of g on x, 12
$\langle \phi \rangle$	the point in M_d^n corresponding to $\phi \in \mathrm{Hom}_d^n$, 25
$G \ll F$	notation for $G(x) \le c_1 F(x) + c_2$, 101
A^G	the ring of invariant of G acting on A, 11
A_g	the moduli space of principally polarized abelian varieties, 88
$\bar{\mathfrak{a}}(\phi)$	ideal class associated to minimal model of ϕ, 49
$\mathrm{Aut}(\phi)$	the automorphism group of the map ϕ, 34
BiCrit_d	set of maps with exactly two critical points, 42
$C_1(n)$	dynamical modular curve for maps $\psi_b(z) = z/(z^2 + b)$, 66
$\deg_X(S)$	minimal degree of polynomial vanishing on S, 98
$e_\phi(P)$	the ramification index of ϕ at P, 91
$e_v(\phi)$	the valuation of the Macaulay resultant of ϕ, 47
$\varepsilon_v(\phi)$	exponent at v of the minimal resultant of ϕ, 48
F_g	moduli space of K3 surfaces, 52
Fix	map from Hom_d to the fixed points of the map, 36
$\Phi_{\phi,n}$	n-dynatomic polynomial, 57
ϕ^f	$= f^{-1} \circ \phi \circ f$, the conjugate of ϕ by f, 4
ϕ^n	nth iterate of the function ϕ, 3
$\phi^{\#}(P)$	chordal derivative of ϕ at P, 108
$\phi_c(z)$	the polynomial $z^2 + c$, 59
G_ϕ	the subgroup of $\mathrm{Gal}(\overline{K}/K)$ such that $\sigma(\phi) \sim \phi$, 116
$\mathbb{G}_m$	the multiplicative group, 15
$\widehat{G}_{\Phi,v}(P)$	Green function for lift Φ of ϕ, 76
$\hat{g}_{\phi,v}$	Green function (local canonical height), 76
$H^1(G, A)$	cohomology set for G acting on A, 115
$\hat{h}_{\mathrm{crit}}(\phi)$	critical canonical height of the map ϕ, 99

$\hat{h}^+_\phi, \hat{h}^-_\phi$	canonical heights for a regular affine automorphism, 83
$\hat{h}^+_\phi, \hat{h}^-_\phi$	canonical heights on a K3 surface, 86
h	the Weil height on $\mathbb{P}^N(\overline{K})$, 69
$h(\phi)$	the height of the map ϕ via $\phi \in \mathrm{Hom}_d^N \subset \mathbb{P}^N$, 70
$(\mathrm{Hom}_d)^{\mathrm{ss}}$	semistable locus in Hom_d, 21
$(\mathrm{Hom}_d)^{\mathrm{s}}$	stable locus in Hom_d, 21
$\mathrm{Hom}_d(\mathbb{P}^1(\mathbb{C}))$	degree d rational self-maps of $\mathbb{P}^1(\mathbb{C})$, 3
Hom_d^n	degree d morphisms $\mathbb{P}^n \to \mathbb{P}^n$, 7
$\mathrm{Hom}_d^n(m)$	maps with a marked point of formal period n, 8
ι_1, ι_2	involutions on the surface $S_{\boldsymbol{A},\boldsymbol{B}}$, 51
$\mathcal{J}^{\mathrm{f}}(\phi)$	the filled Julia set of ϕ, 76
K_ϕ	the field of moduli for ϕ, 116
$L(\phi)$	the Lyapunov exponent of ϕ, 108
L_d	the set of flexible Lattès maps in M_d, 101
$\ell(0) \cdot x$	specialization of the 1-parameter subgroup ℓ, 17
Λ_ϕ^n	the multiplier spectrum of ϕ, 25
$\lambda_\phi(\alpha)$	multiplier of ϕ at the periodic point α, 24
Lat_d	the set of flexible Lattès maps in Hom_d, 101
$(\mathsf{M}_d)^{\mathrm{ss}}$	moduli space of semistable points in Hom_d, 21
$(\mathsf{M}_d)^{\mathrm{s}}$	moduli space of stable points in Hom_d, 21
M_K	complete set of inequivalent normalized absolute values on K, 69
M_K	set of absolute values on the function field $K = k(C)$, 45
M_d	moduli space of self-morphisms of $\mathbb{P}^1$, 8
M_d^n	moduli space of self-morphisms of $\mathbb{P}^n$, 8
$\mathsf{M}_d^n(m)$	moduli space of maps with a marked point of formal period n, 8
$\mathsf{M}_d^{\mathrm{BiCrit}}$	image of BiCrit_d in M_d, 43
$\mathsf{M}_d^{\mathrm{crit}}$	moduli space of degree d maps with marked critical points, 92
$\mathsf{M}_d^{\mathrm{crit}}[i](r,n)$	subvariety of $\mathsf{M}_d^{\mathrm{crit}}$ with marked critical point having specified portrait, 93
$\mathfrak{M}$	the Mandelbrot set, 88
$\overline{\mathsf{M}}_2$	completion of M_2, 33
$\hat{\mu}_\phi$	invariant measure associated to ϕ, 107
$\mu(\phi)$	height expansion coefficient of ϕ, 81
$\mu^{\mathcal{L}}(x,\ell)$	integer invariant attached to 1-parameter subgroup ℓ, 18
$\mathcal{O}_\phi(x)$	the forward orbit of x for the map ϕ, 3
$\mathcal{O}_{\phi,\psi}(P)$	full orbit of P for two maps ϕ and ψ, 80
$\mathsf{P}_d^{\mathrm{crit}}$	subvariety of $\mathsf{M}_d^{\mathrm{crit}}$ corresponding to polynomial maps, 96

$\mathsf{P}_d^{\mathrm{crit}}[i](r,n)$	subvariety of $\mathsf{P}_d^{\mathrm{crit}}$ with point having given critical point orbit, 96
$\mathrm{Per}(\phi, X)$	set of periodic points for ϕ, 4
$\mathrm{Per}_n(\phi, X)$	set of periodic points of period n for ϕ, 4
$\mathrm{Per}_n^*(\phi)$	periodic points of formal period n, 92
$\mathrm{Per}_n^{**}(\phi)$	points of exact period n, 8
PGL_{n+1}	the projective linear group, 8
$\mathrm{PrePer}(\phi, X)$	set of preperiodic points for ϕ, 4
$\mathrm{PrePer}_{m,n}(\phi, X)$	set of preperiodic points of tail m and period n, 4
$\mathrm{PrePer}_{r,n}^*(\phi)$	preperiodic points of tail length r and formal period n, 92
$\mathfrak{R}(\phi)$	the minimal resultant of ϕ, 48
Rat_d^n	degree d rational maps $\mathbb{P}^n \to \mathbb{P}^n$, 7
$(\mathcal{S}\phi)(z)$	the Schwarzian derivative of ϕ, 29
$S_{\boldsymbol{A},\boldsymbol{B}}$	K3 surface determined by the coefficients $\boldsymbol{A}$ and $\boldsymbol{B}$, 50
$\mathcal{S}_n$	symmetric group, 11
$\sigma(g,x)$	the image of the action of g on x, 12
$\sigma_{i,n}(\phi)$	symmetric function of multipliers of ϕ, 25
$\mathrm{Stab}(f)$	the stabilizer of the map f, 12
$\mathcal{T}_P(\mathbb{P}^1)$	tangent space of $\mathbb{P}^1$ at P, 25
$\mathcal{T}_P(\mathbb{P}^N)$	tangent space of $\mathbb{P}^N$ at P, 27
$\mathrm{Twist}_K(\phi)$	the set of K-twists of ϕ, 113
V/G	the quotient of V by the finite group G, 11
$X^{\mathrm{ss}}(\mathcal{L})$	semistable locus, 16
$X^{\mathrm{s}}(\mathcal{L})$	stable locus, 16
$X_0(n)$	smooth projective model for $Y_0(n)$, 60
$X_1(n)$	smooth projective model for $Y_1(n)$, 59
$X_{(0)}^{\mathrm{s}}(\mathcal{L})$	stable locus with dimension 0 stabilizer, 16
$Y_0(n)$	dynamical modular curve, 60
$Y_1(n)$	dynamical modular curve, 59
$Z(\phi)$	locus of indeterminacy of ϕ, 40

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