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Classification and Identification of Lie Algebras

Libor Šnobl
Pavel Winternitz

American Mathematical Society

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Preface

The purpose of this book is to serve as a tool for practitioners of Lie algebra and Lie group theory, i.e., for those who apply Lie algebras and Lie groups to solve problems arising in science and engineering. It is not intended to be a textbook on Lie theory, nor is it oriented towards one specific application, for instance the analysis of symmetries of differential equations. We restrict our attention to finite-dimensional Lie algebras over the fields of complex and real numbers.

In any application Lie algebras typically arise as sets of linear operators that commute with a given operator, say the Hamiltonian of a physical system. Alternatively, Lie groups arise as groups of (local) transformations leaving some object invariant; the corresponding Lie algebra then consists of vector fields generating 1-parameter subgroups. The object may be for instance the set of all solutions of a system of equations. The equations can be differential, difference, algebraic or integral ones, or some combination of such equations. They may be linear or nonlinear. In any case, the Lie algebra is realized by some operators in a basis that is usually not the standard one and that depends crucially on the manner in which it was obtained. The structure constants of Lie algebras can be calculated in any basis, but they in turn are basis dependent and reveal very little about the actual structure of the given Lie algebra.

After the Lie algebra $\mathfrak{g}$ associated with a studied problem is found, the next task that faces the researcher is to identify the Lie algebra as an abstract Lie algebra. In some cases $\mathfrak{g}$ may be isomorphic to a known algebra given in some accessible list. This is certainly the case for semisimple Lie algebras in view of Cartan's classification of all simple Lie algebras over the complex numbers, and subsequent classification of their real forms.

The fundamental Levi theorem, stating that every finite dimensional Lie algebra is isomorphic to a semidirect sum of a semisimple Lie algebra and the maximal solvable ideal (the radical) greatly simplifies the task of identifying a given Lie algebra. The weak link is that no complete classification of solvable Lie algebras exists, nor can one be expected to be produced in the future.

The problem addressed in this book is that of transforming a randomly obtained basis of a Lie algebra into a "canonical basis" in which all basis independent features of the Lie algebra are directly visible. For low dimensional Lie algebras (of dimension less or equal six) this makes it possible to identify the Lie algebra completely. In this book we give a representative list of all such Lie algebras. As stated above, in any dimension a complete identification can be performed for semisimple Lie algebras. We also describe some classes of nilpotent and solvable Lie algebras of arbitrary finite dimensions for which a complete classification exists and hence an exact identification is possible.

The book has four parts. The first presents some general results and concepts that are used in the subsequent chapters. In particular such invariant notions as the dimension of ideals in the characteristic series, and the invariants of the coadjoint representation are introduced.

In Part 2 we present algorithms that accomplish the following tasks:

(1) An algorithm for determining whether the algebra $\mathfrak{g}$ can be decomposed into a direct sum. If $\mathfrak{g}$ is decomposable the algorithm provides a basis in which $\mathfrak{g}$ is explicitly decomposed into a direct sum of indecomposable Lie subalgebras.

(2) A further algorithm is presented to find the radical $R(\mathfrak{g})$ and the Levi factor, i.e., the semisimple component of $\mathfrak{g}$.

(3) If the Lie algebra is solvable, for instance if it is the radical of a larger algebra, then it is necessary to identify its nilradical, i.e., the maximal nilpotent ideal. A rational (i.e., avoiding calculation of eigenvalues) algorithm for performing this is presented.

The text includes many examples illustrating various situations that may arise in such computations. All these algorithms have been implemented on computers.

Part 3 is devoted to solvable and nilpotent Lie algebras. While a complete classification of such algebras seems not to be feasible, it is possible to take a class of nilpotent Lie algebras and construct all extensions of these algebras to solvable ones. Finite-dimensional solvable Lie algebras with Abelian, Heisenberg, Borel, filiform and quasifiliform nilradicals are presented in Part 3.

Part 4 of the book consists of tables of all indecomposable Lie algebras of dimension n where $1 \leq n \leq 6$. They are ordered in such a way as to make the identification of any given low-dimensional Lie algebra written in an arbitrary basis as simple as possible. Any Lie algebra up to dimension 6 is isomorphic to precisely one entry in the tables. Essential characteristics of each algebra including its Casimir invariants are also provided.

The book is based on material that was previously dispersed in journal articles, many of them written by one or both of the authors of this book together with collaborators. The tables in Part 4 are based on older results and have been independently verified, in some cases corrected, unified and ordered by structural properties of the algebras (rather than by the way they were originally obtained).

Libor Šnobl and Pavel Winternitz

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L. Šnobl dedicates this book to his parents Libuše and Zdeněk. P. Winternitz dedicates this book to his wife Milada and his sons Peter and Michael. We both thank them for their support and encouragement.

Bibliography

1. L. Abellanas and L. Martínez Alonso, *A general setting for Casimir invariants*, J. Mathematical Phys. **16** (1975), 1580–1584.
2. ———, *Invariants in enveloping algebras under the action of Lie algebras of derivations*, J. Math. Phys. **20** (1979), no. 3, 437–440.
3. J. M. Ancochea Bermúdez, R. Campoamor-Stursberg, and L. García Vergnolle, *Solvable Lie algebras with naturally graded nilradicals and their invariants*, J. Phys. A **39** (2006), no. 6, 1339–1355.
4. ———, *Indecomposable Lie algebras with nontrivial Levi decomposition cannot have filiform radical*, Int. Math. Forum **1** (2006), no. 5-8, 309–316.
5. R. E. Beck, B. Kolman, and I. N. Stewart, *Computing the structure of a Lie algebra*, Computers in Nonassociative Rings and Algebras (San Antonio, TX, 1976) (R. E. Beck and B. Kolman, eds.), Academic Press, New York, 1977, pp. 167–188.
6. L. Bianchi, *Sugli spazii a tre dimensioni che ammettono un gruppo continuo di movimenti*, Soc. Ital. Sci. Mem. di Mat. **11** (1898), 267–352.
7. V. Boyko, J. Patera, and R. Popovych, *Computation of invariants of Lie algebras by means of moving frames*, J. Phys. A **39** (2006), no. 20, 5749–5762.
8. ———, *Invariants of Lie algebras with fixed structure of nilradicals*, J. Phys. A **40** (2007), no. 1, 113–130.
9. ———, *Invariants of triangular Lie algebras*, J. Phys. A **40** (2007), no. 27, 7557–7572.
10. ———, *Invariants of triangular Lie algebras with one nil-independent diagonal element*, J. Phys. A **40** (2007), no. 32, 9783–9792.
11. ———, *Invariants of solvable Lie algebras with triangular nilradicals and diagonal nilindependent elements*, Linear Algebra Appl. **428** (2008), no. 4, 834–854.
12. N. Burgoyne and R. Cushman, *Conjugacy classes in linear groups*, J. Algebra **44** (1977), no. 2, 339–362.
13. R. Campoamor-Stursberg, *Invariants of solvable rigid Lie algebras up to dimension* 8, J. Phys. A **35** (2002), no. 30, 6293–6306.
14. ———, *Non-semisimple Lie algebras with Levi factor* $\mathfrak{so}(3)$, $\mathfrak{sl}(2,\mathbb{R})$ *and their invariants*, J. Phys. A **36** (2003), no. 5, 1357–1369.
15. ———, *On the invariants of some solvable rigid Lie algebras*, J. Math. Phys. **44** (2003), no. 2, 771–784.
16. ———, *Some remarks concerning the invariants of rank one solvable real Lie algebras*, Algebra Colloq. **12** (2005), no. 3, 497–518.
17. ———, *A note on the classification of nine dimensional Lie algebras with nontrivial Levi decomposition*, Int. Math. Forum **2** (2007), no. 25-28, 1341–1344.
18. ———, *Structural data and invariants of nine dimensional real Lie algebras with nontrivial Levi decomposition*, Nova Science Publishers Inc., New York, 2009.
19. ———, *Solvable Lie algebras with an* $\mathbb{N}$*-graded nilradical of maximal nilpotency degree and their invariants*, J. Phys. A **43** (2010), no. 14, 145202.
20. É. Cartan, *Les groupes réels simples, finis et continus*, Ann. Sci. École Norm. Sup. (3) **31** (1914), 263–355.
21. ———, *Groupes simples clos et ouverts et géométrie riemannienne*, J. Math. Pures Appl. **8** (1929), 1–34.
22. ———, *Sur la structure des groupes des transformations finis et continus*, 2nd ed., Vuibert, Paris, 1933.
23. ———, *La méthode du repère mobile, la théorie des groupes continus et les espaces généralisés*, Actualites Sci. Indust., vol. 194, Hermann, Paris, 1935.

24. ———, *La théorie des groupes finis et continus et la géométrie différentielle traitées par la méthode du repère mobile. Leçons professées à la Sorbonne. Rédigées par Jean Leray*, Cahiers Sci., vol. 18, Gauthier-Villars, Paris, 1937.
25. H. B. G. Casimir, *Über die Konstruktion einer zu den irreduziblen Darstellungen halbeinfacher kontinuierlicher Gruppen gehörigen Differentialgleichung*, Proc. Kon. Akad. Wetensch. **34** (1931), 844–846.
26. H. B. G. Casimir and B. L. van der Waerden, *Algebraischer Beweis der vollständigen Reduzibilität der Darstellungen halbeinfacher Liescher Gruppen*, Math. Ann. **111** (1935), no. 1, 1–12.
27. A. A. Chesnokov, *Symmetries and exact solutions of the rotating shallow-water equations*, European J. Appl. Math. **20** (2009), no. 5, 461–477.
28. C. Chevalley, *Théorie des groupes de Lie*. Tome II: *Groupes algébriques*, Actualités Sci. Ind., vol. 1152, Hermann, Paris, 1951.
29. C. Chevalley and S. Eilenberg, *Cohomology theory of Lie groups and Lie algebras*, Trans. Amer. Math. Soc. **63** (1948), 85–124.
30. A. M. Cohen and W. A. de Graaf, *Lie algebraic computation*, Comput. Phys. Comm. **97** (1996), no. 1-2, 53–62.
31. A. M. Cohen, W. A. de Graaf, and L. Rónyai, *Computations in finite-dimensional Lie algebras*, Discrete Math. Theor. Comput. Sci. **1** (1997), no. 1, 129–138.
32. W. A. de Graaf, *An algorithm for the decomposition of semisimple Lie algebras*, Theoret. Comput. Sci. **187** (1997), no. 1-2, 117–122.
33. ———, *Calculating the structure of a semisimple Lie algebra*, J. Pure Appl. Algebra **117/118** (1997), 319–329.
34. ———, *Lie algebras: theory and algorithms*, North-Holland Math. Library, vol. 56, North-Holland, Amsterdam, 2000.
35. W. A. de Graaf, G. Ivanyos, A. Küronya, and L. Rónyai, *Computing Levi decompositions in Lie algebras*, Appl. Algebra Engrg. Comm. Comput. **8** (1997), no. 4, 291–303.
36. W. A. de Graaf, G. Ivanyos, and L. Rónyai, *Computing Cartan subalgebras of Lie algebras*, Appl. Algebra Engrg. Comm. Comput. **7** (1996), no. 5, 339–349.
37. M. A. del Olmo, M. A. Rodríguez, P. Winternitz, and H. Zassenhaus, *Maximal abelian subalgebras of pseudounitary Lie algebras*, Linear Algebra Appl. **135** (1990), 79–151.
38. A. Di Bucchianico and D. Loeb, *Umbral calculus*, Electron. J. Combin. **Dynamic Surveys** (2000), DS3.
39. L. E. Dickson, *Linear algebras*, Hafner Publishing Co., New York, 1960. Reprinting of Cambridge Tracts in Math. Math. Phys., vol. 16.
40. A. Dimakis, F. Müller-Hoissen, and T. Striker, *Umbral calculus, discretization, and quantum mechanics on a lattice*, J. Phys. A **29** (1996), no. 21, 6861–6876.
41. J. Dixmier, *Enveloping algebras*, Grad. Stud. Math., vol. 11, Amer. Math. Soc., Providence, RI, 1996. Revised reprint of the 1977 translation.
42. D. Ž. Djoković, J. Patera, P. Winternitz, and H. Zassenhaus, *Normal forms of elements of classical real and complex Lie and Jordan algebras*, J. Math. Phys. **24** (1983), no. 6, 1363–1374.
43. E. B. Dynkin, *Classification of the simple Lie groups*, Rec. Math. [Mat. Sbornik] N. S. **18(60)** (1946), no. 3, 347–352 (Russian).
44. ———, *The structure of semi-simple algebras*, Uspehi Matem. Nauk (N.S.) **2** (1947), no. 4(20), 59–127 (Russian); English transl. in Amer. Math. Soc. Translation **1950** (1950), no. 17.
45. F. J. Echarte, J. R. Gómez, and J. Núñez, *Les algèbres de Lie filiformes complexes dérivées d'autres algèbres de Lie*, Lois d'algèbres et variétés algébriques (Colmar, 1991) (M. Goze, ed.), Travaux en Cours, vol. 50, Hermann, Paris, 1996, pp. 45–55.
46. K. Erdmann and M. J. Wildon, *Introduction to Lie algebras*, Springer Undergrad. Math. Ser., Springer, London, 2006.
47. G. Favre, *Une algèbre de Lie caractéristiquement nilpotente de dimension* 7, C. R. Acad. Sci. Paris Sér. A-B **274** (1972), A1338–A1339.
48. M. Fels and P. J. Olver, *Moving coframes*. I: *A practical algorithm*, Acta Appl. Math. **51** (1998), no. 2, 161–213; II: *Regularization and theoretical foundations* **55** (1999), no. 2, 127–208.

49. L. Gagnon and P. Winternitz, *Lie symmetries of a generalised nonlinear Schrödinger equation.* I: *The symmetry group and its subgroups*, J. Phys. A **21** (1988), no. 7, 1493–1511; II: *Exact solutions* **22** (1989), no. 5, 469–497.
50. F. Gantmacher, *On the classification of real simple Lie groups*, Rec. Math. [Mat. Sbornik] N.S. **5 (47)** (1939), 217–250.
51. M.-P. Gong, *Classification of nilpotent Lie algebras of dimension* 7 (*over algebraically closed fields and* $\mathbb{R}$), ProQuest LLC, Ann Arbor, MI, 1998. Ph.D. Thesis, University of Waterloo.
52. M. Goze and Yu. Khakimdjanov, *Nilpotent Lie algebras*, Mathematics Appl., vol. 361, Kluwer, Dordrecht, 1996.
53. ———, *Nilpotent and solvable Lie algebras*, Handbook of Algebra, Vol. 2 (M. Hazewinkel, ed.), North-Holland, Amsterdam, 2000, pp. 615–663.
54. Harish-Chandra, *The characters of semisimple Lie groups*, Trans. Amer. Math. Soc. **83** (1956), 98–163.
55. J. Hietarinta, *Direct methods for the search of the second invariant*, Phys. Rep. **147** (1987), no. 2, 87–154.
56. J. Hrivnák and P. Novotný, *Twisted cocycles of Lie algebras and corresponding invariant functions*, Linear Algebra Appl. **430** (2009), no. 4, 1384–1403.
57. V. Hussin, P. Winternitz, and H. Zassenhaus, *Maximal abelian subalgebras of complex orthogonal Lie algebras*, Linear Algebra Appl. **141** (1990), 183–220.
58. N. Jacobson, *Basic algebra.* I, W. H. Freeman and Co., San Francisco, CA, 1974.
59. ———, *Lie algebras*, Dover Publications Inc., New York, 1979. Republication of the 1962 original.
60. W. Killing, *Die Zusammensetzung der stetigen endlichen Transformationsgruppen.*, Math. Ann. **31** (1888), no. 2, 252–290; **33** (1888), no. 1, 1–48; **34** (1889), no. 1, 57–122; **36** (1890), no. 2, 161–189.
61. A. A. Kirillov, *Elements of the theory of representations*, 2nd ed., "Nauka", Moscow, 1978 (Russian).
62. A. W. Knapp, *Lie groups beyond an introduction*, 2nd ed., Progr. Math., vol. 140, Birkhäuser, Boston, MA, 2002.
63. B. Kostant, *On the conjugacy of real Cartan subalgebras.* I, Proc. Nat. Acad. Sci. U. S. A. **41** (1955), 967–970.
64. M. Krawtchouk, *Über vertauschbare Matrizen*, Rend. Circ. Mat. Palermo (1) **51** (1927), no. 1, 126–130.
65. G. I. Kruchkovich, *Classification of three-dimensional Riemannian spaces according to groups of motions*, Uspehi Matem. Nauk (N.S.) **9** (1954), no. 1(59), 3–40 (Russian).
66. J. M. Lee, *Manifolds and differential geometry*, Grad. Stud. Math., vol. 107, Amer. Math. Soc., Providence, RI, 2009.
67. G. F. Leger and E. M. Luks, *Cohomology of nilradicals of Borel subalgebras*, Trans. Amer. Math. Soc. **195** (1974), 305–316.
68. D. Levi, M. C. Nucci, C. Rogers, and P. Winternitz, *Group theoretical analysis of a rotating shallow liquid in a rigid container*, J. Phys. A **22** (1989), no. 22, 4743–4767.
69. D. Levi, P. Tempesta, and P. Winternitz, *Lorentz and Galilei invariance on lattices*, Phys. Rev. D **69** (2004), no. 10, 105011.
70. ———, *Umbral calculus, difference equations and the discrete Schrödinger equation*, J. Math. Phys. **45** (2004), no. 11, 4077–4105.
71. E. E. Levi, *Sulla struttura dei gruppi finiti e continui*, Atti Accad. Sci. Torino **40** (1905), 551–565.
72. S. Lie, *Theorie der Transformationsgruppen.* Vol. I, B. G. Teubner, Leipzig, 1888; Vol. II, 1890; Vol. III, 1893.
73. A. I. Mal′cev, *On the representation of an algebra as a direct sum of the radical and a semi-simple subalgebra*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **36** (1942), 42–45.
74. ———, *Commutative subalgebras of semi-simple Lie algebras*, Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **9** (1945), 291–300 (Russian); English transl. in Amer. Math. Soc. Translation **1951** (1951), no. 40.
75. ———, *Foundations of linear algebra* (J. B. Roberts, ed.), translated by T. C. Brown, W. H. Freeman, San Francisco, CA – London, 1963.
76. L. Martina, G. Soliani, and P. Winternitz, *Partially invariant solutions of a class of nonlinear Schrödinger equations*, J. Phys. A **25** (1992), no. 16, 4425–4435.

77. V. V. Morozov, *Classification of nilpotent Lie algebras of sixth order*, Izv. Vysš. Učebn. Zaved. Matematika **1958** (1958), no. 4 (5), 161–171 (Russian).
78. G. M. Mubarakzjanov, *On solvable Lie algebras*, Izv. Vysš. Učehn. Zaved. Matematika **1963** (1963), no. no 1 (32), 114–123 (Russian).
79. ———, *Classification of real structures of Lie algebras of fifth order*, Izv. Vysš. Učebn. Zaved. Matematika **1963** (1963), no. 3 (34), 99–106 (Russian).
80. ———, *Classification of solvable Lie algebras of sixth order with a non-nilpotent basis element*, Izv. Vysš. Učebn. Zaved. Matematika **1963** (1963), no. 4 (35), 104–116 (Russian).
81. ———, *Certain theorems on solvable Lie algebras*, Izv. Vysš. Učebn. Zaved. Matematika **1966** (1966), no. 6 (55), 95–98 (Russian).
82. J.-C. Ndogmo, *Invariants of solvable Lie algebras of dimension six*, J. Phys. A **33** (2000), no. 11, 2273–2287.
83. J.-C. Ndogmo and P. Winternitz, *Generalized Casimir operators of solvable Lie algebras with abelian nilradicals*, J. Phys. A **27** (1994), no. 8, 2787–2800.
84. ———, *Solvable Lie algebras with abelian nilradicals*, J. Phys. A **27** (1994), no. 2, 405–423.
85. P. Novotný and J. Hrivnák, *On* (α, β, γ)*-derivations of Lie algebras and corresponding invariant functions*, J. Geom. Phys. **58** (2008), no. 2, 208–217.
86. P. J. Olver, *Applications of Lie groups to differential equations*, Grad. Texts in Math., vol. 107, Springer, New York, 1986.
87. J. Patera, R. T. Sharp, P. Winternitz, and H. Zassenhaus, *Invariants of real low dimension Lie algebras*, J. Mathematical Phys. **17** (1976), no. 6, 986–994.
88. J. Patera, P. Winternitz, and H. Zassenhaus, *The maximal solvable subgroups of the* $\mathrm{SU}(p, q)$ *groups and all subgroups of* $\mathrm{SU}(2, 1)$, J. Mathematical Phys. **15** (1974), 1378–1393.
89. ———, *The maximal solvable subgroups of* $\mathrm{SO}(p, q)$ *groups*, J. Mathematical Phys. **15** (1974), 1932–1938.
90. ———, *Maximal abelian subalgebras of real and complex symplectic Lie algebras*, J. Math. Phys. **24** (1983), no. 8, 1973–1985.
91. J. N. Pecina-Cruz and Y. Ne'eman, *On the calculation of invariants of Lie algebras*, Canad. J. Phys. **72** (1994), no. 7-8, 466–496 (English, with English and French summaries).
92. J. Pedlosky, *Geophysical Fluid Dynamics*, Springer, Providence, RI, 1990.
93. M. Perroud, *The maximal solvable subalgebras of the real classical Lie algebras*, J. Mathematical Phys. **17** (1976), no. 6, 1028–1033.
94. ———, *The fundamental invariants of inhomogeneous classical groups*, J. Math. Phys. **24** (1983), no. 6, 1381–1391.
95. A. Z. Petrov, *Einstein spaces* (J. Woodrow, ed.), translated by R. F. Kelleher, Pergamon Press, Oxford, 1969.
96. C. Quesne, *Giant dipole transitions in the nuclear* $\mathrm{WSp}(6, \mathbb{R})$ *model*, Phys. Lett. B **188** (1987), no. 1, 1–5.
97. ———, *The nuclear collective* $\mathrm{wsp}(6, \mathbb{R})$ *model*, Ann. Physics **185** (1988), no. 1, 46–85.
98. G. Racah, *Group theory and spectroscopy*, Ergeb. Exakt. Naturwiss., vol. 37, Springer, Berlin, 1965, pp. 28–84.
99. A. Ramani, B. Grammaticos, and T. Bountis, *The Painlevé property and singularity analysis of integrable and nonintegrable systems*, Phys. Rep. **180** (1989), no. 3, 159–245.
100. D. W. Rand, *PASCAL programs for identification of Lie algebras.* I: *RADICAL—a program to calculate the radical and nil radical of parameter-free and parameter-dependent Lie algebras*, Comput. Phys. Comm. **41** (1986), no. 1, 105–125; III: *Levi decomposition and canonical basis* **46** (1987), no. 2, 311–322.
101. D. W. Rand, P. Winternitz, and H. Zassenhaus, *PASCAL programs for the identification of Lie algebras.* II: *SPLIT—a program to decompose parameter-free and parameter-dependent Lie algebras into direct sums*, Comput. Phys. Comm. **46** (1987), no. 2, 297–309.
102. ———, *On the identification of a Lie algebra given by its structure constants.* I: *Direct decompositions*, *Levi decompositions*, *and nilradicals*, Linear Algebra Appl. **109** (1988), 197–246.
103. S. M. Roman and G.-C. Rota, *The umbral calculus*, Advances in Math. **27** (1978), no. 2, 95–188.
104. G. Rosensteel and D. J. Rowe, *Nuclear* $\mathrm{Sp}(3, \mathbb{R})$ *model*, Phys. Rev. Lett. **38** (1977), no. 1, 10–14.

105. ______, *On the algebraic formulation of collective models.* III: *The symplectic shell model of collective motion*, Ann. Physics **126** (1980), no. 2, 343–370.
106. J. L. Rubin and P. Winternitz, *Solvable Lie algebras with Heisenberg ideals*, J. Phys. A **26** (1993), no. 5, 1123–1138.
107. È. N. Safiullina, *Classification of nilpotent Lie algebras of order* 7, Candidates' Works (1964), Math., Mech., Phys., Izdat. Kazan. Univ., Kazan, 1964, pp. 66–69 (Russian).
108. H. Samelson, *Notes on Lie algebras*, 2nd ed., Universitext, Springer, New York, 1990.
109. D. H. Sattinger and O. L. Weaver, *Lie groups and algebras with applications to physics, geometry, and mechanics*, Appl. Math. Sci., vol. 61, Springer, New York, 1986.
110. C. Seeley, 7*-dimensional nilpotent Lie algebras*, Trans. Amer. Math. Soc. **335** (1993), no. 2, 479–496.
111. A. Shabanskaya and G. Thompson, *Six-dimensional Lie algebras with a five-dimensional nilradical*, J. Lie Theory **23** (2013), no. 2, 313–355.
112. T. Skjelbred and T. Sund, *Sur la classification des algèbres de Lie nilpotentes*, C. R. Acad. Sci. Paris Sér. A-B **286** (1978), no. 5, A241–A242.
113. L. Šnobl, *On the structure of maximal solvable extensions and of Levi extensions of nilpotent Lie algebras*, J. Phys. A **43** (2010), no. 50, 505202.
114. L. Šnobl, *Maximal solvable extensions of filiform algebras*, Arch. Math. (Brno) **47** (2011), no. 5, 405–414.
115. L. Šnobl and D. Karásek, *Classification of solvable Lie algebras with a given nilradical by means of solvable extensions of its subalgebras*, Linear Algebra Appl. **432** (2010), no. 7, 1836–1850.
116. L. Šnobl and P. Winternitz, *A class of solvable Lie algebras and their Casimir invariants*, J. Phys. A **38** (2005), no. 12, 2687–2700.
117. ______, *All solvable extensions of a class of nilpotent Lie algebras of dimension n and degree of nilpotency $n-1$*, J. Phys. A **42** (2009), no. 10, 105201.
118. ______, *Solvable Lie algebras with Borel nilradicals*, J. Phys. A **45** (2012), no. 9, 095202.
119. J. Sonn and H. Zassenhaus, *On the theorem on the primitive element*, Amer. Math. Monthly **74** (1967), 407–410.
120. H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact solutions of Einstein's field equations*, 2nd ed., Cambridge Monogr. Math. Phys., Cambridge Univ. Press, Cambridge, 2003.
121. M. Sugiura, *Conjugate classes of Cartan subalgebras in real semi-simple Lie algebras*, J. Math. Soc. Japan **11** (1959), 374–434.
122. D. A. Suprunenko and R. I. Tyshkevich, *Commutative matrices*, Academic Press, New York, 1968.
123. S. Tremblay and P. Winternitz, *Solvable Lie algebras with triangular nilradicals*, J. Phys. A **31** (1998), no. 2, 789–806.
124. ______, *Invariants of the nilpotent and solvable triangular Lie algebras*, J. Phys. A **34** (2001), no. 42, 9085–9099.
125. Gr. Tsagas, *Classification of nilpotent Lie algebras of dimension eight*, J. Inst. Math. Comput. Sci. Math. Ser. **12** (1999), no. 3, 179–183.
126. Gr. Tsagas and A. Kobotis, *Nilpotent Lie algebras of dimension nine*, J. Inst. Math. Comput. Sci. Math. Ser. **4** (1991), no. 1, 21–28.
127. Gr. Tsagas, A. Kobotis, and T. Koukouvinos, *Classification of nilpotent Lie algebras of dimension nine whose maximum abelian ideal is of dimension seven*, Int. J. Comput. Math. **74** (2000), no. 1, 5–28.
128. P. Turkowski, *Low-dimensional real Lie algebras*, J. Math. Phys. **29** (1988), no. 10, 2139–2144.
129. ______, *Solvable Lie algebras of dimension six*, J. Math. Phys. **31** (1990), no. 6, 1344–1350.
130. ______, *Structure of real Lie algebras*, Linear Algebra Appl. **171** (1992), 197–212.
131. M. Vergne, *Cohomologie des algèbres de Lie nilpotentes*: *Application à l'étude de la variété des algèbres de Lie nilpotentes*, C. R. Acad. Sci. Paris Sér. A-B **267** (1968), A867–A870.
132. J. Voisin, *On some unitary representations of the Galilei group.* I: *Irreducible representations*, J. Mathematical Phys. **6** (1965), 1519–1529; II: *Two-particle systems*, 1822–1832.
133. Y. Wang, J. Lin, and S. Deng, *Solvable Lie algebras with quasifiliform nilradicals*, Comm. Algebra **36** (2008), no. 11, 4052–4067.
134. J. H. M. Wedderburn, *Lectures on matrices*, Dover Publications Inc., New York, 1964.

135. G. B. Whitham, *Linear and nonlinear waves*, Pure Appl. Math., Wiley, New York, 1974.
136. P. Winternitz, *Subalgebras of Lie algebras*: *Example of* $\mathrm{sl}(3,\mathbb{R})$, Symmetry in Physics (Montréal, QC, 2002) (P. Winternitz, J. Harnad, C. S. Lam, and J. Patera, eds.), CRM Proc. Lecture Notes, vol. 34, Amer. Math. Soc., Providence, RI, 2004, pp. 215–227.
137. P. Winternitz and H. Zassenhaus, *Decomposition theorems for maximal abelian subalgebras of the classical algebras*, Technical Report CRMA-1199, Centre de recherches en mathématiques appliquées, Université de Montréal, Montréal, QC, 1984.
138. H. Zassenhaus, *Über eine Verallgemeinerung des Henselschen Lemmas*, Arch. Math. (Basel) **5** (1954), 317–325.
139. ______, *Lie groups, Lie algebras and representation theory*, Sem. Math. Sup., vol. 75, Presses Univ. Montréal, Montréal, QC, 1981.
140. D. Zwillinger, *Handbook of differential equations*, 2nd ed., Academic Press, Boston, MA, 1992.

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