Volume 34

CRM MONOGRAPH SERIES

Centre de Recherches Mathématiques
Montréal

Function Theory on Symplectic Manifolds

Leonid Polterovich
Daniel Rosen

American Mathematical Society

Function Theory on Symplectic Manifolds

https://doi.org/10.1090/crmm/034

Volume 34

CRM
MONOGRAPH
SERIES

Centre de Recherches Mathématiques
Montréal

Function Theory on Symplectic Manifolds

Leonid Polterovich
Daniel Rosen

The Centre de Recherches Mathématiques (CRM) was created in 1968 to promote research in pure and applied mathematics and related disciplines. Among its activities are special theme years, summer schools, workshops, postdoctoral programs, and publishing. The CRM receives funding from the Natural Sciences and Engineering Research Council (Canada), the FRQNT (Québec), the NSF (USA), and its partner universities (Université de Montréal, McGill, UQAM, Concordia, Université Laval, Université de Sherbrooke and University of Ottawa). It is affiliated with the lnstitut des Sciences Mathématiques (ISM). For more information visit www.crm.math.ca.

American Mathematical Society
Providence, Rhode Island USA

The production of this volume was supported in part by the Fonds de recherche du Québec–Nature et technologies (FRQNT) and the Natural Sciences and Engineering Research Council of Canada (NSERC).

2010 *Mathematics Subject Classification.* Primary 53Dxx; Secondary 57R17, 81S10, 81P15, 22E65, 20F99, 28A10.

For additional information and updates on this book, visit
www.ams.org/bookpages/crmm-34

Library of Congress Cataloging-in-Publication Data

Polterovich, Leonid, 1963– author.
Function theory on symplectic manifolds / Leonid Polterovich, Daniel Rosen.
pages cm. — (CRM monograph series ; 34)
Includes bibliographical references and index.
ISBN 978-1-4704-1693-5 (acid-free paper)
1. Symplectic manifolds. 2. Geometric function theory. 3. Quantum measure theory. I. Rosen, Daniel, 1980- author. II. Title.

QA665.P65 2014
516.3′6—dc23

2014020767

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
Visit the AMS home page at `http://www.ams.org/`

10 9 8 7 6 5 4 3 2 1   19 18 17 16 15 14

Contents

Preface

The symplectic revolution of the 1980s gave rise to the discovery of surprising rigidity phenomena involving symplectic manifolds, their subsets, and their diffeomorphisms. These phenomena have been detected with the help of a variety of novel powerful methods, including Floer theory, a version of Morse theory on the loop spaces of symplectic manifolds. A number of recent advances show that there is yet another manifestation of symplectic rigidity, taking place in function spaces associated to a symplectic manifold. These spaces exhibit unexpected properties and interesting structures, giving rise to an alternative intuition and new tools in symplectic topology, and providing a motivation to study the *function theory on symplectic manifolds*, which forms the subject of the present book.

Recall that a symplectic structure on a $2n$-dimensional manifold M is given by a closed differential 2-form ω which in appropriate local coordinates is given by $\omega = \sum_{j=1}^{n} \mathrm{d}p_j \wedge \mathrm{d}q_j$. The Poisson bracket of a pair of smooth compactly supported functions F, G on M is a canonical operation given by

$$\{F, G\} = \sum_j \left(\frac{\partial F}{\partial q_j} \frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j} \frac{\partial G}{\partial q_j} \right).$$

The Poisson bracket, which is one of our main characters, plays a fundamental role in symplectic geometry and its applications. For instance, it governs Hamiltonian mechanics. The symplectic manifold M serves as the phase space of a mechanical system. The evolution (or Hamiltonian flow) $h_t \colon M \to M$ of the system is determined by its time-dependent energy $H_t \in C^\infty(M)$. Hamilton's famous equation describing the motion of the system is given, in the Heisenberg picture, by $\dot{F}_t = \{F_t, H_t\}$, where $F_t = F \circ h_t$ stands for the time evolution of an observable function F on M under the Hamiltonian flow h_t. The diffeomorphisms h_t coming from all possible energies H_t form a group $\mathrm{Ham}(M, \omega)$, called the group of Hamiltonian diffeomorphisms. For closed simply connected manifolds this group is just the identity component of the symplectomorphism group. The group Ham can be considered as an infinite-dimensional Lie group. The function space $C^\infty(M)$ is, roughly speaking, the Lie algebra of this group, and the Poisson bracket is its Lie bracket.

The structure of the function theory we are going to develop can be illustrated with the help of the following picture. Fix your favorite $t > 0$

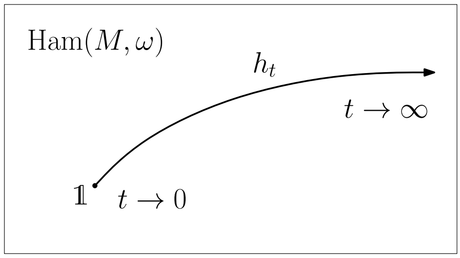

FIGURE 0.1. Two opposite regimes

and consider the natural mapping $C^\infty(M) \to \mathrm{Ham}(M)$ which takes a (time-independent) function H to the time-t map h_t of the corresponding Hamiltonian flow. In principle, this mapping enables one to translate information about Hamiltonian diffeomorphisms (which nowadays is quite a developed subject, see Chapter 4) into the language of function spaces. This naive plan works successfully in two opposite regimes, infinitesimal (when $t \to 0$) and asymptotic (when $t \to \infty$) (see Figure 0.1).

Working in the infinitesimal regime, one arrives at a surprising phenomenon of *C^0-robustness of the Poisson bracket.* Observe that the expression for the Poisson bracket involves the first derivatives of the functions F and G. Nevertheless, the functional $\Phi(F, G) := \|\{F, G\}\|$, where $\|\cdot\|$ stands for the uniform norm of a function, exhibits robustness with respect to C^0-perturbations. In particular, as we shall show in Chapter 2, Φ is lower semi-continuous in the uniform norm. Even though this result sounds analytical in nature, it turns out to be closely related to a remarkable bi-invariant geometry on the group $\mathrm{Ham}(M, \omega)$ discovered by Hofer in 1990. We shall discuss various facets of C^0-robustness of the Poisson bracket. One of them is the *Poisson bracket invariant* of a quadruple of subsets of a symplectic manifold discussed in Chapter 7. Its definition is based on an elementary looking variational problem involving the functional Φ, while its study involves a variety of methods of "hard" symplectic topology. Another facet is symplectic approximation theory, discussed in Chapter 8. Its basic objective is to find an optimal uniform approximation of a given pair of functions by a pair of (almost) Poisson commuting functions.

The asymptotic regime gives rise to the theory of *symplectic quasi-states* presented in Chapter 5. A symplectic quasi-state is a monotone functional $\zeta\colon C^\infty(M) \to \mathbb{R}$ with $\zeta(1) = 1$ which is linear on every Poisson-commutative subalgebra, but not necessarily on the whole function space. The origins of this notion go back to foundations of quantum mechanics and Aarnes' theory of topological quasi-states, an interesting branch of abstract functional analysis. In our context, nonlinear quasi-states on higher-dimensional manifolds are provided by Floer theory, the cornerstone of modern symplectic topology. Interestingly enough, symplectic quasi-states are closely related to *quasi-morphisms* on the group of Hamiltonian diffeomorphisms $\mathrm{Ham}(M, \omega)$.

Roughly speaking, a quasi-morphism on a group is "a homomorphism up to a bounded error." This group-theoretical notion coming from bounded cohomology has been intensively studied in the past decade due to its various applications to geometry and dynamics. We discuss it in Chapter 3. A recent survey of quasi-states and quasi-morphisms in symplectic topology can be found in Entov's ICM-2014 talk [57].

Quasi-states serve as a useful tool for a number of problems in symplectic topology such as symplectic intersections, Hofer's geometry on groups of Hamiltonian diffeomorphisms, and Lagrangian knots. These applications are presented in Chapter 6. In addition, quasi-states provide yet another insight into robustness of the Poisson brackets, see Section 4.6.

Besides applications to some mainstream problems in symplectic topology, function theory on symplectic manifolds opens up a prospect of using "hard" symplectic methods in quantum mechanics. Mathematical quantization and, mostnotably, the quantum-classical correspondence principle provide a tool which enables one to translate basic notions of classical mechanics into quantum language. In general, a meaningful translation of symplectic rigidity phenomena involving subsets and diffeomorphisms faces serious analytical and conceptual difficulties. However, such a translation becomes possible if one shifts the focus from subsets and morphisms of manifolds to function spaces. We present some first steps in this direction in Chapter 9.

The book is a fusion of a research monograph on function theory on symplectic manifolds and an introductory survey of symplectic topology. On the introductory side, the first chapter discusses some basic symplectic constructions and fundamental phenomena, including the Eliashberg–Gromov C^0-rigidity theorem, Arnold's symplectic fixed point conjecture, and Hofer's metric, while in the last three chapters the reader will find an informal crash course on Floer theory. Even though our intention was to make the book as self-contained as possible, the reader is encouraged to consult earlier symplectic literature, such as the classical monographs [107, 108] by McDuff and Salamon. We also refer the reader to the manuscript by Oh [121] on Floer theory. The reader is assumed to have familiarity with basic differential and algebraic topology.

Most of the results presented in the book are based on a number of joint papers by L.P. with Michael Entov. L.P. expresses his gratitude to Michael for long years of pleasant collaboration. Furthermore, some central results of the book are joint with Lev Buhovsky (Poisson bracket invariants and symplectic approximation), Yakov Eliashberg (Lagrangian knots), and Frol Zapolsky (Poisson bracket inequality and rigidity of partitions of unity). L.P. cordially thanks all of them.

Parts of the material have been taught by L.P. in graduate courses at University of Chicago and Tel Aviv University, in a lecture series at UCLA, and (with the assistance of D.R.) in a mini-course at University of Melbourne. We thank these institutions for such an invaluable opportunity. We are indebted to Strom Borman, Lev Buhovsky, François Charette, Adi Dickstein, Michael Entov, Morimichi Kawasaki, Asaf Kislev, Karina Samvelyan, Egor Schelukhin, and Frol Zapolsky for very useful comments on the manuscript, as well as to Dorit Aharonov, Peter Albers, Paul Biran, Paul Busch, Danny Calegari, Yakov Eliashberg, Helmut Hofer, Vincente Humilière, Gil Kalai, Vadim Kaloshin, Yael Karshon, Guy Kindler, Dusa McDuff, Yong-Geun Oh, Yaron Ostrover, Iosif Polterovich, Victor Polterovich, Dietmar Salamon, Felix Schlenk, Paul Seidel, Sobhan Seyfaddini, Ivan Smith, Misha Sodin, Shmuel Weinberger, Amie Wilkinson, and Jinxin Xue for illuminating discussions on various topics of the book. Additionally, we thank the anonymous referees for very helpful critical remarks.

We are indebted to Andrei Iacob for a superb copyediting and very useful critical comments and suggestions on the presentation, and to Carl Sperber for correcting a number of grammatical and stylistic errors.

Finally, special thanks go to François Lalonde, who encouraged us to write this book.

This research has been partially supported by the National Science Foundation grant DMS-1006610, the Israel Science Foundation grants 509/07 and 178/13, as well as the European Research Council Advanced grant 338809.

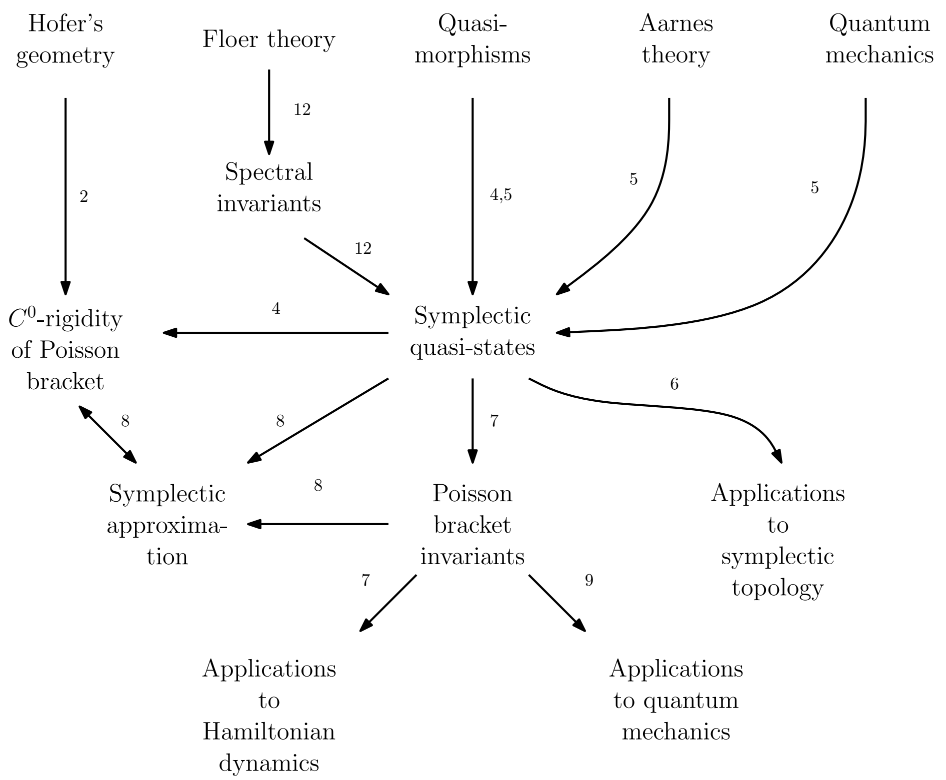

FIGURE 0.2. Subject road map. Numbers next to arrows indicate relevant chapters.

Bibliography

1. J. F. Aarnes, *Quasi-states and quasi-measures*, Adv. Math. **86** (1991), no. 1, 41–67.
2. M. Akveld and D. Salamon, *Loops of Lagrangian submanifolds and pseudoholomorphic discs*, Geom. Funct. Anal. **11** (2001), no. 4, 609–650.
3. S. T. Ali, C. Carmeli, T. Heinosaari, and A. Toigo, *Commutative POVMs and fuzzy observables*, Found. Phys. **39** (2009), no. 6, 593–612.
4. V. I. Arnold, *Sur une propriété topologique des applications globalement canoniques de la mécanique classique*, C. R. Acad. Sci. Paris **261** (1965), 3719–3722.
5. ______, *On a characteristic class entering into conditions of quantization* (Russian), Funkcional. Anal. i Prilozen. **1** (1967), 1–14.
6. ______, *A stability problem and ergodic properties of classical dynamical systems* (Russian), Proc. Internat. Congr. Math. (Moscow, 1966), Izdat. "Mir", Moscow, 1968, pp. 387–392.
7. ______, *Mathematical methods of classical mechanics*, translated by K. Vogtmann and A. Weinstein, Grad. Texts in Math., vol. 60, Springer, New York, 1978.
8. ______, *The first steps of symplectic topology* (Russian), Uspekhi Mat. Nauk **41** (1986), no. 6(252), 3–18, 229; English transl., Russian Math. Surveys **41** (1986), no. 6, 1–21.
9. E. Arthurs and M. S. Goodman, *Quantum correlations: a generalized Heisenberg uncertainty relation*, Phys. Rev. Lett. **60** (1988), no. 24, 2447–2449.
10. E. Arthurs and J. L. Kelly Jr., *On the simultaneous measurement of a pair of conjugate observables*, Bell System Tech. J. **44** (1965), 725–729.
11. M. F. Atiyah, *Convexity and commuting Hamiltonians*, Bull. London Math. Soc. **14** (1982), no. 1, 1–15.
12. M. Audin and M. Damian, *Théorie de Morse et homologie de Floer*, Savoirs Actuels (Les Ulis), EDP Sciences, Les Ulis; CNRS Éditions, Paris, 2010.
13. A. Banyaga, *Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique*, Comment. Math. Helv. **53** (1978), no. 2, 174–227.
14. A. Banyaga and D. Hurtubise, *Lectures on Morse homology*, Kluwer Texts Math. Sci., vol. 29, Kluwer Acad. Publ., Dordrecht, 2004.
15. J. Barge and É. Ghys, *Surfaces et cohomologie bornée*, Invent. Math. **92** (1988), no. 3, 509–526.
16. A. Barvinok, *A course in convexity*, Grad. Stud. Math., vol. 54, Amer. Math. Soc., Providence, RI, 2002.
17. C. Bavard, *Longueur stable des commutateurs*, Enseign. Math. (2) **37** (1991), no. 1-2, 109–150.
18. J. S. Bell, *On the problem of hidden variables in quantum mechanics*, Rev. Modern Phys. **38** (1966), 447–452.
19. G. Ben Simon, *The nonlinear Maslov index and the Calabi homomorphism*, Commun. Contemp. Math. **9** (2007), no. 6, 769–780.
20. S. K. Berberian, *Notes on spectral theory*, Van Nostrand Mathematical Studies, vol. 5, Van Nostrand, Princeton, NJ–Toronto, ON–London, 1966.

21. F. A. Berezin, *General concept of quantization*, Comm. Math. Phys. **40** (1975), 153–174.
22. P. Biran, *A stability property of symplectic packing*, Invent. Math. **136** (1999), no. 1, 123–155.
23. ______, *From symplectic packing to algebraic geometry and back*, European Congress of Mathematics, Vol. II (Barcelona, 2000) (C. Casacuberta, R. M. Miró-Roig, J. Verdera, and S. Xambó-Descamps, eds.), Progr. Math., vol. 202, Birkhäuser, Basel, 2001, pp. 507–524.
24. P. Biran, M. Entov, and L. Polterovich, *Calabi quasimorphisms for the symplectic ball*, Commun. Contemp. Math. **6** (2004), no. 5, 793–802.
25. M. Bordemann, E. Meinrenken, and M. Schlichenmaier, *Toeplitz quantization of Kähler manifolds and* gl(N), $N \to \infty$ *limits*, Comm. Math. Phys. **165** (1994), no. 2, 281–296.
26. M. S. Borman, *Symplectic reduction of quasi-morphisms and quasi-states*, J. Symplectic Geom. **10** (2012), no. 2, 225–246.
27. ______, *Quasi-states, quasi-morphisms, and the moment map*, Int. Math. Res. Not. IMRN **11** (2013), 2497–2533.
28. M. S. Borman and F. Zapolsky, *Quasi-morphisms on contactomorphism groups and contact rigidity*, Geom. Topol., to appear, available at `arXiv:1308.3224`.
29. D. Borthwick and A. Uribe, *Almost complex structures and geometric quantization*, Math. Res. Lett. **3** (1996), no. 6, 845–861.
30. R. Bott and L. W. Tu, *Differential forms in algebraic topology*, Grad. Texts in Math., vol. 82, Springer, New York, 1982.
31. M. Brandenbursky, *Bi-invariant metrics and quasi-morphisms on groups of Hamiltonian diffeomorphisms of surfaces*, available at `arXiv:1306.3350`.
32. L. Buhovsky, *The* $2/3$*-convergence rate for the Poisson bracket*, Geom. Funct. Anal. **19** (2010), no. 6, 1620–1649.
33. ______, *The gap between near commutativity and almost commutativity in symplectic category*, Electron. Res. Announc. Math. Sci. **20** (2013), 71–76.
34. L. Buhovsky, M. Entov, and L. Polterovich, *Poisson brackets and symplectic invariants*, Selecta Math. (N.S.) **18** (2012), no. 1, 89–157.
35. L. Buhovsky and Y. Ostrover, *On the uniqueness of Hofer's geometry*, Geom. Funct. Anal. **21** (2011), no. 6, 1296–1330.
36. D. Burago, S. Ivanov, and L. Polterovich, *Conjugation-invariant norms on groups of geometric origin*, Groups of Diffeomorphisms (R. Penner, D. Kotschick, T. Tsuboi, N. Kawazumi, T. Kitano, and Y. Mitsumatsu, eds.), Adv. Stud. Pure Math., vol. 52, Math. Soc. Japan, Tokyo, 2008, pp. 221–250.
37. P. Busch, M. Grabowski, and P. J. Lahti, *Operational quantum physics*, Lecture Notes in Phys. New Ser. m Monogr., vol. 31, Springer, Berlin, 1995.
38. P. Busch, T. Heinonen, and P. Lahti, *Noise and disturbance in quantum measurement*, Phys. Lett. A **320** (2004), no. 4, 261–270.
39. ______, *Heisenberg's uncertainty principle*, Phys. Rep. **452** (2007), no. 6, 155–176.
40. P. Busch, P. Lahti, and R. F. Werner, *Noise operators and measures of RMS error and disturbance in quantum mechanics*, available at `arXiv:1312.4393`.
41. E. Calabi, *On the group of automorphisms of a symplectic manifold*, Problems in Analysis (Princeton, NJ, 1969) (R. C. Gunning, ed.), Princeton Univ. Press, Princeton, NJ, 1970, pp. 1–26.
42. D. Calegari, *scl*, MSJ Memoirs, vol. 20, Math. Soc. Japan, Tokyo, 2009.
43. F. Cardin and S. Vazzoler, *Symplectic rigidity and weak commutativity*, J. Symplectic Geom. **11** (2013), no. 2, 163–166.
44. F. Cardin and C. Viterbo, *Commuting Hamiltonians and Hamilton-Jacobi multi-time equations*, Duke Math. J. **144** (2008), no. 2, 235–284.

45. J. F. Cariñena, F. Falceto, and M. F. Rañada, *A geometric approach to a generalized virial theorem*, J. Phys. A **45** (2012), no. 39, 395210, 19.
46. Yu. V. Chekanov, *Lagrangian tori in a symplectic vector space and global symplectomorphisms*, Math. Z. **223** (1996), no. 4, 547–559.
47. ______, *Invariant Finsler metrics on the space of Lagrangian embeddings*, Math. Z. **234** (2000), no. 3, 605–619.
48. C. C. Conley and E. Zehnder, *The Birkhoff–Lewis fixed point theorem and a conjecture of V. I. Arnold*, Invent. Math. **73** (1983), no. 1, 33–49.
49. ______, *Morse-type index theory for flows and periodic solutions for Hamiltonian equations*, Comm. Pure Appl. Math. **37** (1984), no. 2, 207–253.
50. A. Ya. Dubovicki, *On differentiable mappings of an n-dimensional cube into a k-dimensional cube* (Russian), Mat. Sbornik N.S. **32(74)** (1953), 443–464.
51. D. Eisenbud, U. Hirsch, and W. Neumann, *Transverse foliations of Seifert bundles and self-homeomorphism of the circle*, Comment. Math. Helv. **56** (1981), no. 4, 638–660.
52. Ya. M. Eliashberg, *A theorem on the structure of wave fronts and its application in symplectic topology* (Russian), Funktsional. Anal. i Prilozhen. **21** (1987), no. 3, 65–72, 96.
53. Ya. M. Eliashberg and L. Polterovich, *Bi-invariant metrics on the group of Hamiltonian diffeomorphisms*, Internat. J. Math. **4** (1993), no. 5, 727–738.
54. ______, *Local Lagrangian 2-knots are trivial*, Ann. of Math. (2) **144** (1996), no. 1, 61–76.
55. ______, *Symplectic quasi-states on the quadric surface and Lagrangian submanifolds*, available at `arXiv:1006.2501`.
56. M. Entov, *Commutator length of symplectomorphisms*, Comment. Math. Helv. **79** (2004), no. 1, 58–104.
57. ______, *Quasi-morphisms and quasi-states in symplectic topology*, Proceedings of the ICM (Seoul, 2014), to appear.
58. M. Entov and L. Polterovich, *Calabi quasimorphism and quantum homology*, Int. Math. Res. Not. **30** (2003), 1635–1676.
59. ______, *Quasi-states and symplectic intersections*, Comment. Math. Helv. **81** (2006), no. 1, 75–99.
60. ______, *Symplectic quasi-states and semi-simplicity of quantum homology*, Toric Topology (M. Harada, Y. Karshon, M. Masuda, and T. Panov, eds.), Contemp. Math., vol. 460, Amer. Math. Soc., Providence, RI, 2008, pp. 47–70.
61. ______, *C^0-rigidity of the double Poisson bracket*, Int. Math. Res. Not. IMRN **6** (2009), 1134–1158.
62. ______, *Lie quasi-states*, J. Lie Theory **19** (2009), no. 3, 613–637.
63. ______, *Rigid subsets of symplectic manifolds*, Compos. Math. **145** (2009), no. 3, 773–826.
64. ______, *C^0-rigidity of Poisson brackets*, Symplectic Topology and Measure Preserving Dynamical Systems (Snowbird, UT, 2007) (A. Fathi, Y.-G. Oh, and C. Viterbo, eds.), Contemp. Math., vol. 512, Amer. Math. Soc., Providence, RI, 2010, pp. 25–32.
65. M. Entov, L. Polterovich, and D. Rosen, *Poisson brackets, quasi-states and symplectic integrators*, Discrete Contin. Dyn. Syst. **28** (2010), no. 4, 1455–1468.
66. M. Entov, L. Polterovich, and F. Zapolsky, *Quasi-morphisms and the Poisson bracket*, Pure Appl. Math. Q. **3** (2007), no. 4, 1037–1055.
67. A. Floer, *Symplectic fixed points and holomorphic spheres*, Comm. Math. Phys. **120** (1989), no. 4, 575–611.
68. A. Floer and H. Hofer, *Coherent orientations for periodic orbit problems in symplectic geometry*, Math. Z. **212** (1993), no. 1, 13–38.

69. K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Toric degeneration and nondisplaceable Lagrangian tori in $S^2 \times S^2$*, Int. Math. Res. Not. IMRN **13** (2012), 2942–2993.
70. ———, *Spectral invariants with bulk, quasimorphisms and Lagrangian Floer theory*, available at `arXiv:1105.5124`.
71. J.-M. Gambaudo and É. Ghys, *Commutators and diffeomorphisms of surfaces*, Ergodic Theory Dynam. Systems **24** (2004), no. 5, 1591–1617.
72. É. Ghys, *Knots and dynamics*, International Congress of Mathematicians. Vol. I (Madrid, 2006) (M. Sanz-Solé, J. Soria, J. L. Varona, and J. Verdera, eds.), Eur. Math. Soc., Zürich, 2007, pp. 247–277.
73. A. B. Givental, *Nonlinear generalization of the Maslov index*, Theory of Singularities and Its Applications (V. I. Arnold, ed.), Adv. Soviet Math., vol. 1, Amer. Math. Soc., Providence, RI, 1990, pp. 71–103.
74. ———, *The nonlinear Maslov index*, Geometry of Low-Dimensional Manifolds. 2 (Durham, 1989) (S. K. Donaldson and C. B. Thomas, eds.), London Math. Soc. Lecture Note Ser., vol. 151, Cambridge Univ. Press, Cambridge, 1990, pp. 35–43.
75. A. M. Gleason, *Measures on the closed subspaces of a Hilbert space*, J. Math. Mech. **6** (1957), 885–893.
76. M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), no. 2, 307–347.
77. ———, *Partial differential relations*, Ergeb. Math. Grenzgeb. (3), vol. 9, Springer, Berlin, 1986.
78. D. J. Grubb and T. LaBerge, *Additivity of quasi-measures*, Proc. Amer. Math. Soc. **126** (1998), no. 10, 3007–3012.
79. ———, *Spaces of quasi-measures*, Canad. Math. Bull. **42** (1999), no. 3, 291–297.
80. D. Gu and S. S. Rappaport, *Mobile user registration in cellular systems with overlapping location areas*, 1999 IEEE 49th Vehicular Technology Conference. Vol. 1 (Houston, TX, 1999), IEEE Press, Piscataway, NJ, pp. 802–806.
81. V. Guillemin, *Star products on compact pre-quantizable symplectic manifolds*, Lett. Math. Phys. **35** (1995), no. 1, 85–89.
82. V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), no. 3, 491–513.
83. M. Hayashi, *Quantum information: An introduction*, Springer, Berlin, 2006.
84. M. W. Hirsch, *Differential topology*, Vol. 33, Springer, New York, 1976. Grad. Texts in Math.
85. H. Hofer, *On the topological properties of symplectic maps*, Proc. Roy. Soc. Edinburgh Sect. A **115** (1990), no. 1-2, 25–38.
86. ———, *Estimates for the energy of a symplectic map*, Comment. Math. Helv. **68** (1993), no. 1, 48–72.
87. M. Hutchings, *Recent progress on symplectic embedding problems in four dimensions*, Proc. Natl. Acad. Sci. USA **108** (2011), no. 20, 8093–8099.
88. ———, *Lecture notes on Morse homology* (*with an eye towards Floer theory and pseudo-holomorphic curves*), available at `http://www.math.umass.edu/~sullivan/797SG/hutchings-morse.pdf`.
89. S. Ishikawa, *Uncertainty relations in simultaneous measurements for arbitrary observables*, Rep. Math. Phys. **29** (1991), no. 3, 257–273.
90. B. Janssens, *Unifying decoherence and the Heisenberg principle*, available at `arXiv:quant-ph/0606093`.
91. A. Jenová and S. Pulmannová, *Characterizations of commutative POV measures*, Found. Phys. **39** (2009), no. 6, 613–624.
92. M. Khanevsky, *Hofer's metric on the space of diameters*, J. Topol. Anal. **1** (2009), no. 4, 407–416.
93. N. Kornechuk, *Exact constants in approximation theory*, translated by K. Ivanov, Encyclopedia Math. Appl., vol. 38, Cambridge Univ. Press, Cambridge, 1991.

94. F. Lalonde, *Energy and capacities in symplectic topology*, Geometric Topology (Athens, GA, 1993) (W. H. Kazez, ed.), AMS/IP Stud. Adv. Math., vol. 2, Amer. Math. Soc., Providence, RI, 1997, pp. 328–374.
95. F. Lalonde and D. McDuff, *The geometry of symplectic energy*, Ann. of Math. (2) **141** (1995), no. 2, 349–371.
96. ______, *Hofer's L^∞-geometry: energy and stability of Hamiltonian flows.* I, Invent. Math. **122** (1995), no. 1, 1–33; II, 35–69.
97. N. P. Landsman, *Mathematical topics between classical and quantum mechanics*, Springer Monogr. Math., Springer, New York, 1998.
98. S. Lang, *Algebra*, 3rd ed., Grad. Texts in Math., vol. 211, Springer, New York, 2002.
99. X. Ma and G. Marinescu, *Toeplitz operators on symplectic manifolds*, J. Geom. Anal. **18** (2008), no. 2, 565–611.
100. J. Marsden and A. Weinstein, *Reduction of symplectic manifolds with symmetry*, Rep. Mathematical Phys. **5** (1974), no. 1, 121–130.
101. H. Martens and W. M. de Muynck, *Nonideal quantum measurements*, Found. Phys. **20** (1990), no. 3, 255–281.
102. S. Massar, *Uncertainty relations for positive-operator-valued measures*, Phys. Rev. A (3) **76** (2007), no. 4, 042114, 5.
103. D. McDuff, *Geometric variants of the Hofer norm*, J. Symplectic Geom. **1** (2002), no. 2, 197–252.
104. ______, *Hamiltonian S^1-manifolds are uniruled*, Duke Math. J. **146** (2009), no. 3, 449–507.
105. ______, *Loops in the Hamiltonian group: a survey*, Symplectic Topology and Measure Preserving Dynamical Systems (Snowbird, UT, 2007) (A. Fathi, Y.-G. Oh, and C. Viterbo, eds.), Contemp. Math., vol. 512, Amer. Math. Soc., Providence, RI, 2010, pp. 127–148.
106. ______, *Monodromy in Hamiltonian Floer theory*, Comment. Math. Helv. **85** (2010), no. 1, 95–133.
107. D. McDuff and D. Salamon, *Introduction to symplectic topology*, 2nd ed., Oxford Math. Monogr., Oxford Univ. Press, New York, 1998.
108. ______, *J-holomorphic curves and symplectic topology*, 2nd ed., Amer. Math. Soc. Colloq. Publ., vol. 52, American Mathematical Society, Providence, RI, 2012.
109. J. Milnor, *Morse theory*, Ann. of Math. Stud., vol. 51, Princeton Univ. Press, Princeton, NJ, 1963.
110. J. W. Milnor and J. D. Stasheff, *Characteristic classes*, Ann. of Math. Stud., vol. 76, Princeton Univ. Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1974.
111. B. Misra, *Nonequilibrium entropy, Lyapounov variables, and ergodic properties of classical systems*, Proc. Nat. Acad. Sci. U.S.A. **75** (1978), no. 4, 1627–1631.
112. T. Miyadera and H. Imai, *Heisenberg's uncertainty principle for simultaneous measurement of positive-operator-valued measures*, Phys. Rev. A (3) **78** (2008), no. 5, 052119.
113. N. Monod, *An invitation to bounded cohomology*, International Congress of Mathematicians. Vol. II (Madrid, 2006) (M. Sanz-Solé, J. Soria, J. L. Varona, and J. Verdera, eds.), Eur. Math. Soc., Zürich, 2006, pp. 1183–1211.
114. J. Moser, *On the volume elements on a manifold*, Trans. Amer. Math. Soc. **120** (1965), 286–294.
115. R. Narasimhan, *Analysis on real and complex manifolds*, Adv. Stud. Pure Math., vol. 1, Masson, Paris; North-Holland, Amsterdam, 1968.
116. M. A. Neumark, *On a representation of additive operator set functions*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **41** (1943), 359–361.
117. Y.-G. Oh, *Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks.* II: $(\mathbb{C}\mathrm{P}^n, \mathbb{R}\mathrm{P}^n)$, Comm. Pure Appl. Math. **46** (1993), no. 7, 995–1012.

118. ______, *Chain level Floer theory and Hofer's geometry of the Hamiltonian diffeomorphism group*, Asian J. Math. **6** (2002), no. 4, 579–624.
119. ______, *Construction of spectral invariants of Hamiltonian paths on closed symplectic manifolds*, The Breadth of Symplectic and Poisson Geometry (J. E. Marsden and T. S. Ratiu, eds.), Progr. Math., vol. 232, Birkhäuser Boston, Boston, MA, 2005, pp. 525–570.
120. ______, *Lectures on Floer theory and spectral invariants of Hamiltonian flows*, Morse Theoretic Methods in Nonlinear Analysis and in Symplectic Topology (Montréal, QC, 2004) (P. Biran, O. Cornea, and F. Lalonde, eds.), NATO Sci. Ser. II Math. Phys. Chem., vol. 217, Springer, Dordrecht, 2006, pp. 321–416.
121. ______, *Symplectic topology and Floer homology*, available at `http://www.math.wisc.edu/~oh/all.pdf`.
122. K. Ono, *Floer–Novikov cohomology and the flux conjecture*, Geom. Funct. Anal. **16** (2006), no. 5, 981–1020.
123. Y. Ostrover, *A comparison of Hofer's metrics on Hamiltonian diffeomorphisms and Lagrangian submanifolds*, Commun. Contemp. Math. **5** (2003), no. 5, 803–811.
124. ______, *Calabi quasi-morphisms for some non-monotone symplectic manifolds*, Algebr. Geom. Topol. **6** (2006), 405–434 (electronic).
125. Y. Ostrover and I. Tyomkin, *On the quantum homology algebra of toric Fano manifolds*, Selecta Math. (N.S.) **15** (2009), no. 1, 121–149.
126. M. Ozawa, *Uncertainty relations for joint measurements of noncommuting observables*, Phys. Lett. A **320** (2004), no. 5-6, 367–374.
127. ______, *Universal uncertainty principle in the measurement operator formalism*, J. Opt. B Quantum Semiclass. Opt. **7** (2005), no. 12, S672–S681.
128. C. Pearcy and A. Shields, *Almost commuting matrices*, J. Funct. Anal. **33** (1979), no. 3, 332–338.
129. A. Peres, *Quantum theory: concepts and methods*, Fundam. Theor. Phys., vol. 57, Kluwer, Dordrecht, 1993.
130. S. Piunikhin, D. Salamon, and M. Schwarz, *Symplectic Floer–Donaldson theory and quantum cohomology*, Contact and Symplectic Geometry (Cambridge, 1994) (C. B. Thomas, ed.), Publ. Newton Inst., vol. 8, Cambridge Univ. Press, Cambridge, 1996, pp. 171–200.
131. L. Polterovich, *Symplectic displacement energy for Lagrangian submanifolds*, Ergodic Theory Dynam. Systems **13** (1993), no. 2, 357–367.
132. ______, *The geometry of the group of symplectic diffeomorphisms*, Lectures Math. ETH Zurich, Birkhäuser, Basel, 2001.
133. ______, *Quantum unsharpness and symplectic rigidity*, Lett. Math. Phys. **102** (2012), no. 3, 245–264.
134. ______, *Symplectic geometry of quantum noise*, Comm. Math. Phys. **327** (2014), no. 2, 481–519.
135. L. Polterovich and Z. Rudnick, *Stable mixing for cat maps and quasi-morphisms of the modular group*, Ergodic Theory Dynam. Systems **24** (2004), no. 2, 609–619.
136. L. Polterovich and K. F. Siburg, *On the asymptotic geometry of area-preserving maps*, Math. Res. Lett. **7** (2000), no. 2-3, 233–243.
137. P. Py, *Quasi-morphismes de Calabi et graphe de Reeb sur le tore*, C. R. Math. Acad. Sci. Paris **343** (2006), no. 5, 323–328.
138. ______, *Quasi-morphismes et invariant de Calabi*, Ann. Sci. École Norm. Sup. (4) **39** (2006), no. 1, 177–195.
139. J. Robbin and D. Salamon, *The Maslov index for paths*, Topology **32** (1993), no. 4, 827–844.
140. O. Rousseau, *Quelques propriétés du complexe de Morse-Novikov*, M.Sc. Thesis, Université de Montréal, 2005.

141. Yu. B. Rudyak and F. Schlenk, *Minimal atlases of closed symplectic manifolds*, Commun. Contemp. Math. **9** (2007), no. 6, 811–855.
142. M. Schlichenmaier, *Berezin–Toeplitz quantization for compact Kähler manifolds. A review of results*, Adv. Math. Phys. (2010), Art. ID 927280, 38.
143. M. Schwarz, *Morse homology*, Progr. Math., vol. 111, Birkhäuser, Basel, 1993.
144. ———, *On the action spectrum for closed symplectically aspherical manifolds*, Pacific J. Math. **193** (2000), no. 2, 419–461.
145. S. Seyfaddini, *C^0-limits of Hamiltonian paths and the Oh–Schwarz spectral invariants*, Int. Math. Res. Not. IMRN **21** (2013), 4920–4960.
146. D. Shakhmatov, unpublished.
147. E. Shelukhin, *The action homomorphism, quasimorphisms and moment maps on the space of compatible almost complex structures*, Comment. Math. Helv. **89** (2014), no. 1, 69–123.
148. A. I. Shtern, *The Kazhdan–Milman problem for semisimple compact Lie groups* (Russian), Uspekhi Mat. Nauk **62** (2007), no. 1(373), 123–190; English transl., Russian Math. Surveys **62** (2007), no. 1, 113–174.
149. J.-C. Sikorav, *Systèmes hamiltoniens et topologie symplectique*, Universitá di Pisa, ETS Editrice, Pisa, 1990.
150. S. Smale, *Diffeomorphisms of the 2-sphere*, Proc. Amer. Math. Soc. **10** (1959), 621–626.
151. W. Stulpe, S. Gudder, and J. Hagler, *An uncertainty relation for joint position-momentum measurements*, Found. Phys. Lett. **1** (1988), no. 3, 287–292.
152. T. Sunada, *Riemannian coverings and isospectral manifolds*, Ann. of Math. (2) **121** (1985), no. 1, 169–186.
153. B. Sz.-Nagy, *Extensions of linear transformations in Hilbert space which extend beyond this space*, translated by L. F. Boron, appendix to F. Riesz and B. Sz.-Nagy, Functional Analysis, Frederick Ungar Publishing Co., New York, 1960.
154. M. Usher, *Spectral numbers in Floer theories*, Compos. Math. **144** (2008), no. 6, 1581–1592.
155. ———, *Duality in filtered Floer–Novikov complexes*, J. Topol. Anal. **2** (2010), no. 2, 233–258.
156. ———, *Deformed Hamiltonian Floer theory, capacity estimates and Calabi quasimorphisms*, Geom. Topol. **15** (2011), no. 3, 1313–1417.
157. C. Viterbo, *Symplectic topology as the geometry of generating functions*, Math. Ann. **292** (1992), no. 4, 685–710.
158. C. Voisin, *Hodge theory and complex algebraic geometry.* I, translated by L. Schneps, Cambridge Stud. Adv. Math, vol. 76, Cambridge Univ. Press, Cambridge, 2002.
159. J. von Neumann, *Mathematical foundations of quantum mechanics*, translated by R. T. Beyer, Princeton Univ. Press, Princeton, NJ, 1955.
160. C. Wendl, *Lectures on holomorphic curves in symplectic and contact geometry*, available at `arXiv:1011.1690`.
161. R. F. Wheeler, *Quasi-measures and dimension theory*, Topology Appl. **66** (1995), no. 1, 75–92.
162. J. A. Yorke, *Periods of periodic solutions and the Lipschitz constant*, Proc. Amer. Math. Soc. **22** (1969), 509–512.
163. F. Zapolsky, *Quasi-states and the Poisson bracket on surfaces*, J. Mod. Dyn. **1** (2007), no. 3, 465–475.
164. ———, *On almost Poisson commutativity in dimension two*, Electron. Res. Announc. Math. Sci. **17** (2010), 155–160.
165. ———, *Isotopy-invariant topological measures on closed orientable surfaces of higher genus*, Math. Z. **270** (2012), no. 1-2, 133–143.

Notation Index

Subject Index

Name Index

This is a book on symplectic topology, a rapidly developing field of mathematics which originated as a geometric tool for problems of classical mechanics. Since the 1980s, powerful methods such as Gromov's pseudo-holomorphic curves and Morse-Floer theory on loop spaces gave rise to the discovery of unexpected symplectic phenomena. The present book focuses on function spaces associated with a symplectic manifold. A number of recent advances show that these spaces exhibit intriguing properties and structures, giving rise to an alternative intuition and new tools in symplectic topology. The book provides an essentially self-contained introduction into these developments along with applications to symplectic topology, algebra and geometry of symplectomorphism groups, Hamiltonian dynamics and quantum mechanics. It will appeal to researchers and students from the graduate level onwards.

I like the spirit of this book. It formulates concepts clearly and explains the relationship between them. The subject matter is important and interesting.

—Dusa McDuff, Barnard College, Columbia University

This is a very important book, coming at the right moment. The book is a remarkable mix of introductory chapters and research topics at the very forefront of actual research. It is full of cross fertilizations of different theories, and will be useful to Ph.D. students and researchers in symplectic geometry as well as to many researchers in other fields (geometric group theory, functional analysis, mathematical quantum mechanics). It is also perfectly suited for a Ph.D.-students seminar.

—Felix Schlenk, Université de Neuchâtel

ISBN 978-1-4704-1693-5

9 781470 416935

CRMM/34

For additional information and updates on this book, visit

www.ams.org/bookpages/crmm-34