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Centre de Recherches Mathématiques
Université de Montréal

Asymptotic Methods in Mechanics

Rémi Vaillancourt
Andrei L. Smirnov
Editors

American Mathematical Society

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Asymptotic Methods in Mechanics

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“To our understanding wives,
Luba and Jeanne”

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Foreword

These Proceedings contain papers on mechanics (mostly for thin structures) essentially using the methods of asymptotic integration and asymptotic approaches in the choice of the mathematical models.

Only some of the problems arising in practice have exact solutions. Thus, one is usually forced to limit oneself to approximate solutions, among which one can distinguish the analytical and the numerical solutions.

In the beginning of this century, the analytical solutions took the main place, but lately, especially during the last decades, due to the advent of powerful computers and well-developed numerical methods, in particular, the finite element methods, the situation changed dramatically. Nowadays, most problems can be solved by means of numerical methods. Consequently, the approximate analytical solutions have been unjustly underestimated.

The analytical and numerical solutions have to complement each other. The approximate analytical solutions, among which the asymptotic solutions occupy a central place, clarify qualitatively the solution behaviour and provide a first approximation within a reasonable error. Note that the accuracy of the asymptotic solutions increases as the difficulties in the construction of the numerical solution increase. This is the case, for example, as the plate or shell relative thickness decreases.

These Proceedings consist of two parts.

The first part is a survey of the asymptotic methods applied to the analysis of thin-wall structures. It contains only the main results and a vast bibliography, to which the reader may refer for a detailed study.

The properties of the asymptotic expansions, including the asymptotic expansions of solutions with a parameter, the Laplace method, and the stationary phase and saddle point methods are also discussed.

In this survey, asymptotic expansions of both the singularly perturbed differential equations and the corresponding boundary value problems are analyzed. Both ordinary and partial differential equations are examined. For the linear equations, the WKB approximations and their generalizations to the case of turning points or transition lines (caustics) are considered and Vishik–Lyusternik's method is described.

Then the application of the asymptotic integration methods to free vibrations and buckling of thin shells is considered. Methods developed by Gol′denveizer, Maslov, and Wasow are discussed and numerous references to works containing applications of these methods to problems of shell theory are given.

On the other hand, the one-dimensional beam equation and two-dimensional shells equations themselves are the results of the reductions of three-dimensional

problems of the theory of elasticity to one- or two-dimensional problems in the narrow zones. In this review the asymptotic algorithm of such reduction is briefly discussed.

In the last part of the survey, the singular degeneracy of nonlinear boundary value problems is considered. The algorithm of Vasil'eva–Butuzov and the method of Lyapunov–Schmidt are described.

The second part of these Proceedings contains original papers on mechanics in which an asymptotic approach is used.

The main group of papers [**19, 58, 59, 129, 188, 240, 241**] is devoted to the application of asymptotic methods to thin shell analysis.

In [**240**], the nonlinear axisymmetric deformation of a shell of revolution is considered. The effect of the load intensity on the asymptotic solution is examined.

In [**129**], the buckling of a convex shell of revolution under nonaxisymmetric loading is studied. The buckling mode localized in neighbourhoods of the edges is constructed.

In [**58, 59**], low-frequency free vibrations of a cylindrical shell with a slanted edge and connected shells are considered. The asymptotic solutions, which take into account the influence of the boundary conditions, are obtained for the free frequencies and vibration modes.

In [**189**], the acoustic problem for the vibrations of two concentric cylindrical shells in air is examined. The ratio of the shell radii for which the vibration mode is localized near the excitation zone is found.

Papers [**19**] and [**241**] contain a review of the authors' work on the asymptotic integration of thin shell equations. These papers were presented as reports at the First European Congress of Mathematics in July of 1992 in Paris.

Similar to paper [**189**], the papers [**9**], [**20**], [**47**], [**48**], [**152**] and [**192**] are concerned with the equations describing the behaviour of three-dimensional continuous media.

In [**9**] the perturbation method is used for the construction of an approximate solution to an axisymmetric electromagnetic problem which occurs in the diagnosis of material defects by nondestructive eddy current testing.

In [**20, 47, 48**], [**152**] and [**192**], one- and two-dimensional models for thin structures are used.

In [**47**] and [**48**], a coupled problem for thin flexible profiles moving in an ideal incompressible flow is considered. The case of two flexible profiles moving in an ideal incompressible flow, the interconnection of which is provided by the flow and the vortex traces behind the back edges of the profiles and the case of a flexible airfoil encountering a gust in an incompressible flow, are analyzed.

Papers [**20**] and [**152**] are concerned with the determination of the temperature deformations of metallic mirrors. In these papers the real structures are modeled by means of a system of one-dimensional plasto-elastic beams ([**152**]) or two-dimensional elastic thin plates ([**20**]), respectively. Numerical studies reveal the good quality of the proposed models.

In paper [**192**], the dynamic buckling of a rotating anisotropic shaft is considered. The asymptotic method is used to study the stability of a thin anisotropic shaft with a crack. This problem is very important for an early diagnosis is of a crack.

The general and main aim of these Proceedings demonstrate the importance of

asymptotic methods and simplified asymptotic models and their possible applications to practical problems of mechanics.

Petr E. Tovstik

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Preface

These Proceedings are the result of the scientific cooperation of researchers from the Laboratory of Applied Mechanics of the Institute of Mathematics and Mechanics at St. Petersburg State University, the Departments of Theoretical and Applied Mechanics and the Department of Hydroelasticity of the Faculty of Mathematics and Mechanics of the same University, the Department of Acoustics of the St. Petersburg Institute of Cinema Engineering, the Department of Applied Mathematics of the Riga Technical University and the Department of Mathematics of the University of Ottawa.

The contents are papers on mechanics (mostly on mechanics of thin structures) in which, basically, the problems are formulated by means of an asymptotic approach and the solutions are obtained by the methods of asymptotic integration.

Initially, it was planned to limit the proceedings to original papers containing new results. These papers are presented in the second part of this collection. However, as the papers were being collected, it was realized that, for the convenience of the reader, it would be useful to include a survey on the asymptotic methods in mechanics which are used in these papers. This survey, which comprises the first part of this book, may also serve as an introduction to the field.

A special feature of the collection is the vast bibliography on the applications of asymptotic methods to problems of mechanics, including Russian literature on asymptotic methods, which is generally translated in English, but still not widely known, despite its high quality. The editors made every effort to include the number of pages, in parentheses, for the monographs listed in the bibliography to the Survey, because many of the Russian books may not be easily available to the Western reader and it was thought that this extra piece of information would be useful.

The book is intended for researchers and engineers working in the analysis and construction of thin-walled structures and continuous media, and applied mathematicians, who are interested in asymptotic methods in problems of mechanics. The book would also be useful to advanced undergraduate and postgraduate students in Mathematics, Physics and Engineering.

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada under grant A7691, the Centre de recherches mathématiques of the Université de Montréal, the Ministry of Science, Higher Education and Technical Policy of Russia under the theme #01910056114, and the Institute of Mathematics and Mechanics of St. Petersburg State University.

The opportunity given to the first editor to work in Professor J. A. Goldak's Laboratory in Mechanical Engineering, at Carleton University, through the Scientific Exchange Program between Carleton University and St. Petersburg State

University under the direction of Professor D. R. F. Taylor, has been very helpful in the final stage of preparation of the book.

The editors would like to underline the special role of Professor P. E. Tovstik, who initiated the study of asymptotic methods applied to problems of solid mechanics at St. Petersburg (then Leningrad) State University more than 25 years ago and who lectured to more than half of the contributors.

The editors did the major translation and edition work for the book. Our friend, Dr. A. A. Kolyshkin, while visiting Ottawa, did the formidable task of inserting all the corrections needed in the first revisions of the text. Thanks are due Mr. N. Filippov (St. Petersburg State University) for the excellent bitmap drawings, which were used as templates, by the editors, for the final postcript figures. A very special thank is due to Mrs. V. Sergeeva (St. Petersburg State University) who typeset the text in $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX.

Andrei L. Smirnov and Rémi Vaillancourt

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