Volume 31

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Centre de Recherches Mathématiques
Université de Montréal

Isomonodromic Deformations and Applications in Physics

CRM Workshop
May 1–6, 2000
Montréal, Canada

John Harnad
Alexander Its
Editors

American Mathematical Society

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Isomonodromic Deformations and Applications in Physics

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The Centre de Recherches Mathématiques (CRM) of the Université de Montréal was created in 1968 to promote research in pure and applied mathematics and related disciplines. Among its activities are special theme years, summer schools, workshops, postdoctoral programs, and publishing. The CRM is supported by the Université de Montréal, the Province of Québec (FCAR), and the Natural Sciences and Engineering Research Council of Canada. It is affiliated with the Institut des Sciences Mathématiques (ISM) of Montréal, whose constituent members are Concordia University, McGill University, the Université de Montréal, the Université du Québec à Montréal, and the Ecole Polytechnique. The CRM may be reached on the Web at www.crm.umontreal.ca.

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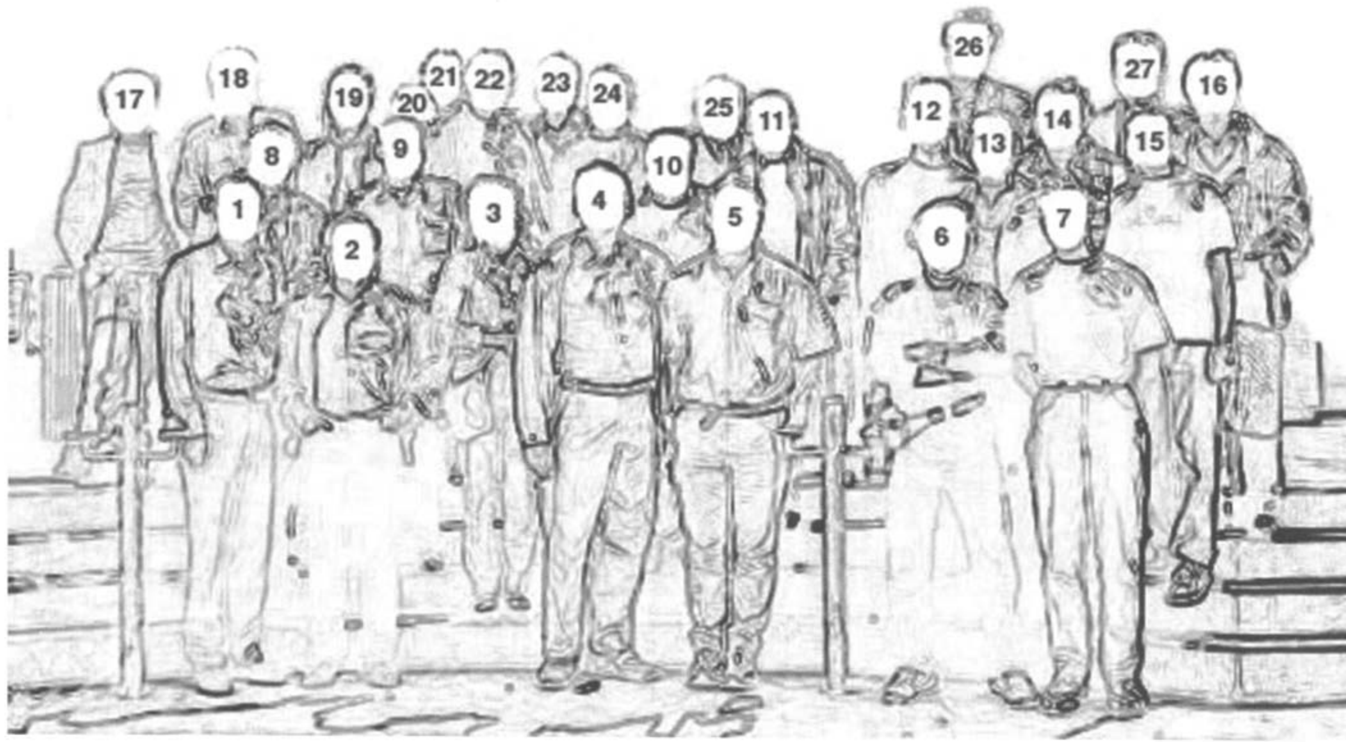
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Preface

This volume represents the first attempt in the current literature to elucidate the important role that isomonodromic deformations play in the theory of integrable systems and their applications in physics. Over the last two decades this area, which is still known to the general mathematical community as the *Inverse Scattering Transform Method* or *Soliton Theory*, has evolved in a vast variety of exciting new algebraic and analytic directions and has found numerous new applications. The methods and applications range from quantum group theory and exactly solvable statistical models to random matrices, random permutations, and number theory. The theory of isomonodromic deformations of systems of differential equations with rational coefficients, and most notably the related apparatus of the Riemann–Hilbert problem, underlie the analytic side of this striking development.

All the contributions in this volume are based on lectures given by the authors at the workshop "Isomonodromic Deformations and Applications in Physics" which took place at the Centre de recherches mathématiques, Université de Montréal, May 1–6, 2000. Each contribution deals to a certain extent with both topics, i.e., with the theory of isomonodromic deformations, as well as with its modern applications. Nevertheless, we decided that it would be to the reader's benefit to split the volume into two parts. The first part, which is called *Isomonodromic Deformations*, represents *mostly* the mathematical aspects of isomonodromic deformations. The second part, called *Applications in Physics and Related Topics*, deals *mostly* with the various appearances of isomonodromic deformation and Riemann–Hilbert methods in the theory of exactly solvable quantum field theory and statistical mechanical models, and related issues.

The first part begins with an expository article by A. Bolibruch which gives a mathematical introduction to the theory of isomonodromic deformations. It also presents an essentially simplified version of Malgrange's proof of the Painlevé property for the generalized Schlesinger equations. The paper by J. Harnad shows how the Fredholm determinants of certain integral operators appearing in the spectral theory of random matrices may be interpreted as isomonodromic tau functions and how to derive the bilinear differential equations they satisfy from the dynamical systems equations of Tracy and Widom. The next paper, by A. Kapaev, deals with the question of classification of all nonequivalent Lax pairs corresponding to a given Painlevé equation. In the paper of D. Korotkin a class of inverse monodromy problems explicitly solvable in terms of Riemann theta-functions is studied. In particular, the τ-functions of the corresponding solutions of the Schlesinger equations are identified with the determinants of Cauchy–Riemann operators acting on spinors defined on hyperelliptic curves.

Explicit solutions to the Schlesinger equations are also discussed in the paper by Y. Ohyama. It is shown that under certain symmetry conditions the Schlesinger equations can be solved by generalized hypergeometric functions. Ohyama's paper is followed by the article of M. Olshanetsky who suggests a generalized isomonodromy deformation theory on algebraic curves in which the moduli of generalized complex structures appear as the deformation parameters. It is shown that the relevant quasi-classical limit leads to Hitchin's algebraically integrable systems (similarly to the usual case of the punctured Riemann sphere, in which the quasi-classical limit of the Schlesinger equations leads to integrable systems of the type governing finite gap solutions in soliton theory). This part of the volume is completed by the expository paper of C. Tracy and H. Widom which explains the appearance of the first nontrivial cases of isomonodromic deformation equations, i.e., the Painlevé equations, in the theory of random matrices. The principal ingredient of the "integrable systems approach" to random matrix theory—the theory of *integrable Fredholm operators*—is outlined as well. This article also provides a bridge between the two parts of the book.

The first paper in the second part of the volume, by M. Bertola, deals with another recent application of isomonodromic deformations—this time in the theory of Frobenius manifolds. It uses the apparatus of Jacobi groups to generalize the previously known polynomial solutions of the WDVV equations to the elliptic case. In addition, an application of Jacobi forms to the study of WZNW models on the torus is discussed. The article of P. Clarkson and C. Cosgrove demonstrates the use of Lie's group-theoretic methods for the study of integrable ordinary differential equations beyond the Painlevé class. These equations include the Chazy equation, which is the classical example of a third order ODE whose solutions possess a movable natural boundary. The paper by F. Göhmann uses the Riemann–Hilbert approach to the theory of integrable systems for evaluation of the asymptotics of correlation functions of the one-dimensional impenetrable electron gas, which plays the role of a universal model for low temperature physics of one-dimensional electrons. In the next, joint paper by F. Göhmann and V. Korepin, an explicit solution of the quantum inverse scattering problem is found for a general case of fundamental graded spin models. The article by Y. Nakamura deals with Toda-type equations, and involves an analysis of Hankel and Toeplitz determinants via continued fractions techniques. This topic is closely related to the random matrix theory discussed in the paper of C. Tracy and H. Widom and to the theory of Schur functions which is the subject of the next contribution, by A. Orlov and D. Scherbin. In this paper the KP τ-function is (a) written in terms of Schur functions, (b) represented as a certain determinant, and (c) written in terms of Milne's hypergeometric functions. As a result, new determinant and integral formulae for certain series of Schur functions are obtained. Palmer's paper presents a proof of the Luther–Peschel formula for the short distance asymptotics of the $2n$ point correlation function of the Ising model below the critical temperature. The method is based on an explicit solution of the relevant inverse monodromy problem, which in turn is shown to be equivalent to the evaluation of the Green's function of a Dirac operator on the punctured complex plane. Finally, the paper of N. Slavnov gives a representation of the partition function of the six vertex model as the Fredholm determinant of an integrable kernel involving Laguerre polynomials. This representation puts this model in the same mathematical context as the integrable random matrix model problems described in the papers of C. Tracy and H. Widom and J. Harnad. The representation is also used to analyze the thermodynamics of the model.

The participants at this workshop benefited from the excellent talks given by the speakers as well as from the many informal discussions these inspired. We thank all those who helped to make the workshop a success and hope that these proceedings will be a useful resource for anyone interested in the fascinating area of isomonodromic deformations and its many applications.

John Harnad and Alexander Its, workshop organizers.

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