

Volume 40

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PROCEEDINGS &
LECTURE NOTES

Centre de Recherches Mathématiques
Université de Montréal

Perspectives
in Riemannian
Geometry

Vestislav Apostolov
Andrew Dancer
Nigel Hitchin
McKenzie Wang
Editors



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American Mathematical Society
Providence, Rhode Island USA

The production of this volume was supported in part by the Fonds pour la Formation de Chercheurs et l'Aide à la Recherche (Fonds FCAR) and the Natural Sciences and Engineering Research Council of Canada (NSERC).

2000 *Mathematics Subject Classification*. Primary 53Cxx, 53Bxx, 53C26;
Secondary 53Dxx, 32Qxx.

Library of Congress Cataloging-in-Publication Data

Perspectives in Riemannian geometry / Vestislav Apostolov . . . [et al.], editors.

p. cm. — (CRM proceedings & lecture notes, ISSN 1065-8580 ; v. 40)

Includes bibliographical references.

ISBN 0-8218-3852-0 (alk. paper)

1. Geometry, Riemannian. 2. Riemannian manifolds. 3. Geometry, Algebraic. I. Apostolov, Vestislav, 1971- II. Title. III. Series.

QA649.P384 2006
516.3'73—dc22

2006042818

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This volume was submitted to the American Mathematical Society
in camera ready form by the Centre de Recherches Mathématiques.

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10 9 8 7 6 5 4 3 2 1 11 10 09 08 07 06

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Preface

The papers in this volume are written by some of the participants of the Short Program on Riemannian Geometry held in June–July 2004 in Montréal. This event was a part of the general scientific program of the Centre de Recherches Mathématiques in Montréal during the summer of 2004. The three-week program opened with a week of introductory short courses, followed by a two-week workshop, and thus gathered close to one hundred participants.

Here is a brief description of the volume contents. Highlighting the main topics of the Proceedings are the three papers which are based on the lectures given during the introductory short courses by Professors M. Anderson, K. Grove and N. J. Hitchin.

The first is Michael Anderson’s comprehensive survey of recent results on the existence of Einstein metrics on open manifolds with a certain structure at infinity. Asymptotically, the simplest structures are those of constant curvature, and this paper covers in great detail global aspects of the hyperbolic (or conformally-compact) case, including applications of this theory to the AdS/CFT correspondence in physics. Another paper complementing this topic is Biquard’s survey of Einstein metrics with asymptotic structure at infinity modeled on the complex hyperbolic space, and their application to CR geometry.

Karsten Grove’s lecture notes survey comparison geometry. The reader will find in it an exhaustive account of the subject, mainly in the framework of lower bounds of sectional curvatures, starting from the basic comparison theorems of Myers, Synge, Rauch, Alexandrov, Toponogov and Bishop, then discussing important developments in conjunction with Morse theory and convexity, the Gromov–Hausdorff topology on spaces of Riemannian manifolds, the geometry of singular spaces, and some recent results in the presence of symmetries. Riemannian metrics with bounded geometry play a central rôle in A. Nabutovsky’s paper which surveys the relationship between the space of Riemannian structures on a closed manifold, computability theory and algorithmic information theory.

What is special about 6, 7 and 8 dimensions? Why do we study Calabi–Yau threefolds, G_2 and $\text{Spin}(7)$ manifolds? These are the questions addressed in Nigel Hitchin’s lecture notes (written with the help of Marco Gualtieri) on these special geometries of great interest in string theory. Based on the fundamental principle of looking at the geometry of open orbits of Lie groups, they present an original entrance into the exciting world of generalized geometry. Another two papers in the volume are concerned with special geometric structures on differentiable manifolds. R. Bryant’s paper provides a systematic study of a class of special Lagrangian submanifolds in \mathbb{C}^n . In their paper, A. Dancer and M. Wang explain their Hamiltonian

approach to cohomogeneity-one Einstein metrics, which leads to new examples and in special cases links to integrable systems.

Three of the papers are devoted to the interaction between Riemannian and complex geometry. The paper by C. Boyer and K. Galicki is a comprehensive survey of new Sasaki–Einstein metrics build out of Kähler–Einstein orbifolds. L. David and P. Gauduchon’s paper provides a thorough study of Bochner-flat Kähler orbifolds from the view point of the CR geometry of the standard sphere, building on an idea of S. Webster. C. LeBrun’s paper studies stability of complex curves-with-boundary in the twistor spaces appearing in the spectacular new approach to Zoll manifolds and split-signature pseudo-Riemannian manifolds that he and L. J. Mason have recently found.

We hope that the contributions presented in this volume reflect to a certain extent the variety of topics presented during the workshop. We are grateful to the Centre de Recherches Mathématiques for their kind and efficient assistance during the workshop and in the preparation of this volume and to Arthur Greenspoon for a careful reading of the manuscripts.

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Montréal, 2006

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This volume presents ten contributions written by participants of the “Short Program on Riemannian Geometry,” a workshop held at the CRM in Montréal in 2004. It will be a valuable reference for graduate students and research mathematicians alike.

ISBN 0-8218-3852-0



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