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CRM PROCEEDINGS & LECTURE NOTES

Centre de Recherches Mathématiques
Université de Montréal

Additive Combinatorics

Andrew Granville
Melvyn B. Nathanson
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Editors

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Preface

Andrew Granville, Melvyn B. Nathanson, and József Solymosi

ABSTRACT. From March 30th to April 5th, 2006, a CRM – Clay School on *Additive Combinatorics* was held at the Université de Montréal, followed by a workshop from April 6th to April 12th, all as part of the 2005 – 2006 special year program, *Analysis in Number Theory*. The school was attended by roughly one hundred participants from sixteen countries around the world, the workshop by forty more people. The first part of this volume contains written versions of most of the lectures given at the school; the second half submitted contributions from the speakers at the workshop.

One of the most active areas in analysis today is the rapidly emerging new topic of "additive combinatorics." Building on Gowers' use of the Freĭman – Ruzsa theorem in harmonic analysis (in particular, his proof of Szemerédi's theorem), Green and Tao famously proved that there are arbitrarily long arithmetic progressions of primes, and Bourgain has given non-trivial estimates for hitherto untouchably short exponential sums. This new subject brings together ideas from harmonic analysis, ergodic theory, discrete geometry, combinatorics, graph theory, group theory, probability theory and number theory to prove some extraordinary results. The basis of the subject is not too difficult: it can be best described as the theory of adding together sets of numbers; in particular understanding the structure of the two original sets if their sum is small. Ideas from all of the above areas come in when providing proofs of key results like the Freĭman – Ruzsa theorem, and the Balog – Gowers – Szemerédi lemma.

Because the background is so broad, the school and conference attracted an eclectic mix of participants, bringing different skills and perspectives. It seems evident that this combustible mixture will continue to lead to exciting advances for some time to come.

The lectures at the school began with some elementary topics:

Andrew Granville: *The basics of additive combinatorics*; *The Freĭman – Ruzsa theorem*; *Uniform distribution and Roth's theorem.*

Józef Solymosi: *Combinatorial discrete geometry and additive combinatorics*; *a proof of Roth's theorem.*

Antal Balog: *The Balog – Szemerédi – Gowers Theorem.*

We would like to thank the Centre de recherches mathématiques, NSERC (Canada), the National Science Foundation (USA), the Clay Mathematics Institute (USA), Dimatia (Czech Republic) and the Université de Montréal, for their generous and willing support of our school and workshop.

The main lecturers, who each gave a series of talks on a deep central theme, were

Ben Green: *Quadratic Fourier analysis.*

Bryna Kra: *Ergodic methods in combinatorial number theory.*

Terry Tao: *Combinatorial and ergodic techniques for proving Szemerédi-type theorems.*

Van Vu: *Structure of sumsets and applications.*

The school ended with lectures by several of the key figures in the early development of what we now call "additive combinatorics":

Imre Ruzsa: *Plünnecke and the others.*

Gregory Freiman: *Inverse additive number theory: Results and problems.*

Endre Szemerédi: *Another proof of Roth's theorem.*

The main lecture series of the school was extraordinarily successful. All of these lecturers made their talks highly accessible, which was reflected in high attendance throughout the meeting, and the buoyant atmosphere. We believe that the lecturers have brought that attitude to the write-ups of their contributions herein! We would particularly like to thank all of the lecturers for their superb talks, and the generosity with which they worked with the participants, helping them to understand this challenging material.

Many leading figures in additive combinatorics, ergodic theory, combinatorics, number theory and harmonic analysis arrived for the workshop, including Jean Bourgain, Tim Gowers, Mei-Chu Chang, Ron Graham, Trevor Wooley, Michael Lacey, Sanju Velani, Sergei Konyagin, Jaroslav Nesetril and many more. In addition, there were two beautiful lectures for the general public: in the first week, Terry Tao gave us further insights into *Long arithmetic progressions in the primes*; in the second week Manjul Bhargava gave a beguiling introduction to his work with Jonathan Hanke on *The representation of integers by quadratic forms.* There were a lot of announcements of exciting new work in the meeting on this very hot topic, most notably perhaps Green and Tao's announcement that they have a viable plan to extend their result on primes in arithmetic progressions to prove a "weak form" of the prime k-tuplets conjecture; and work by Helfgott, and then by Bourgain and Gambaud on the (non-abelian) group generation problem.

This was just the third workshop on additive combinatorics (the first was at the American Institute of Mathematics in September 2004, the second at the University of Bristol in September 2005). Over the next few years there will be several major programs in additive combinatorics and related areas, including during Fall 2007 at the Institute for Advanced Study in Princeton, during Spring 2008 at the Fields Institute in Toronto, and during Fall 2008 at the Mathematical Sciences Research Institute in Berkeley, California.

The meeting would not have been possible without the organizational skills of Louis Pelletier and his team from the Centre de recherches mathématiques, for which we are very grateful.

This book is split into three parts: the proceedings from the school, articles on open questions in the subject, and new research. We begin with basic articles, setting the scene, by the organizers, along with an explicit discussion of the Balog-Szemerédi-Gowers Theorem by Antal Balog, an old but unpublished proof of Roth's theorem by Szemerédi, and a discussion of Bourgain's bounds on exponential sums by Pär Kurlberg.

Andrew Granville: *An Introduction to Additive Combinatorics.*
Józef Solymosi: *Elementary Additive Combinatorics.*
Antal Balog: *Many Additive Quadruples.*
Endre Szemerédi: *An Old New Proof of Roth's Theorem.*
Pär Kurlberg: *Bounds on Exponential Sums over Small Multiplicative Subgroups.*

Next we have three lecture series describing some of the most exciting ideas in the current development of the subject.

Ben Green: *Quadratic Fourier analysis.*
Bryna Kra: *Ergodic Methods in Combinatorial Number Theory.*
Terence Tao: *The Ergodic and Combinatorial Approaches to Szemerédi's Theorem.*

These are followed by several articles highlighting open questions in different aspects of additive combinatorics.

Imre Z. Ruzsa: *Cardinality Questions about Sumsets.*
Ernest S. Croot III and Seva Lev: *Open Problems in Additive Combinatorics.*
Mei-Chu Chang: *Some Problems Related to Sum-Product Theorems.*
Javier Cilleruelo and Andrew Granville: *Lattice Points on Circles, Squares in Arithmetic Progressions and Sumsets of Squares.*
Melvyn B. Nathanson: *Problems in Additive Number Theory.* I.

This is followed by several research articles corresponding to the proceedings of the workshop.

Katalin Gyarmati, Sergei Konyagin and Imre Z. Ruzsa: *Double and Triple Sums Modulo a Prime.*
A. A. Glibichuk and Sergei Konyagin: *Additive Properties of Product Sets in Fields of Prime Order.*
Greg Martin and Kevin O'Bryant: *Many Sets Have More Sums Than Differences.*
Gautami Bhowmik and Jan-Christoph Schlage-Puchta: *Davenport's Constant for Groups of the Form* $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}$.
S. D. Adhikari, R. Balasubramanian and P. Rath: *Some Combinatorial Group Invariants and Their Generalizations with Weights.*

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