Volume 46

CRM PROCEEDINGS & LECTURE NOTES

Centre de Recherches Mathématiques
Université de Montréal

Anatomy of Integers

Jean-Marie De Koninck
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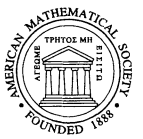
The Centre de Recherches Mathématiques (CRM) of the Université de Montréal was created in 1968 to promote research in pure and applied mathematics and related disciplines. Among its activities are special theme years, summer schools, workshops, postdoctoral programs, and publishing. The CRM is supported by the Université de Montréal, the Province of Québec (FQRNT), and the Natural Sciences and Engineering Research Council of Canada. It is affiliated with the Institut des Sciences Mathématiques (ISM) of Montréal, whose constituent members are Concordia University, McGill University, the Université de Montréal, the Université du Québec à Montréal, and the École Polytechnique. The CRM may be reached on the Web at www.crm.umontreal.ca.



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Contents

What is anatomy?

From March 13th to 17th, 2006, a CRM workshop on the *Anatomy of Integers* was held at the Université de Montréal, as part of the 2005-2006 special year program, *Analysis in Number Theory*. The workshop was attended by roughly ninety participants from seventeen countries. The volume contains surveys based on lectures and lecture series given at the workshop, as well as some contributed papers.

Multiplicative number theory can be roughly split into two great areas – the study of L-functions and their analytic properties, and the study of the prime factors of integers and multiplicative and additive functions on integers. Much of the study of the latter area depends on the details of the size and quantity of the prime factors of integers, or good bounds on the number of integers with different properties (for example, those with only large prime factors). One only has to look at the work of Paul Erdős or Srinivasa Ramanujan to see many examples of how such an intimate understanding can lead to fascinating and unexpected consequences.

Recently there has been much beautiful work in revealing the details of the composition of prime factors of integers by Hildebrand and Tenenbaum, and of their divisors by Ford. Sieve theory can be considered to be part of this quest to understand the detailed structure of the multiplicative properties of integers, and this includes the recent exciting work of Goldston, Pintz and Yildirim.

In mid-March 2006, ninety or so researchers from around the world gathered together at the Centre de recherches mathématiques in Montréal to discuss these interesting topics. The following leading researchers each gave lecture series on their exciting recent works in this area:

Kevin Ford: *Counting numbers with a divisor in a given interval.*
Dan Goldston: *Small Prime Gaps.*
Carl Pomerance: *Anatomy of Euler's function and its friends.*
K. Soundararajan: *Pretentious characters and the Polya–Vinogradov theorem. Smooth numbers.*

There were many other invited talks besides these. This volume reflects some of the lectures at the meeting, and includes interesting articles on a wide variety of subjects connected by the central theme. The first half of the book corresponds to the plenary lectures at the meeting, the second half to contributed talks and papers.

We would like to thank the Centre de recherches mathématiques, NSERC (Canada), FQRNT (Québec), the National Science Foundation (USA), and the Université de Montréal, for their generous and willing support of our workshop.

Anatomy (`a-nat-o-my`) noun: *The science of the shape and structure of an organism and the interrelation of its various parts; a separating or dividing into parts for detailed examination.*

At this meeting we introduced a word, *anatomy*, that we believe best articulates the scope of this large part of multiplicative number theory. Mathematicians rarely introduce new terminology – there is too much already and if everyone continually introduces more then the literature will become unreadable. However the time is occasionally ripe for new nomenclature. Gerald Tenenbaum has long sought to introduce the excellent word

Friable (`fri-a-ble`) adjective: *Easily crumbled or broken; crumble into small pieces,*

in place of the overused and inadequate word

Smooth (`sm-oo-th`) adjective: *Free from projections or unevenness of surface; not rough; of uniform consistency; easy and uniform,*

to describe integers with only small prime factors. Unfortunately the latter is too deeply embedded throughout the literature to be easily dislodged (and "friable" is a more common word in French than in English), which is why it may be best to introduce "anatomy of integers" now and to define its technical meaning from forthcoming developments in the subject.

We hope that these proceedings will contribute to the development of our subject, the *Anatomy of Integers.*

This meeting would not have been possible without the organizational skills of Louis Pelletier and his team from the Centre de Recherches Mathématiques, for which we are very grateful.

Jean-Marie De Koninck
Andrew Granville and
Florian Luca

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Titles in This Series

46 **Jean-Marie De Koninck, Andrew Granville, and Florian Luca, Editors,** Anatomy of integers, 2008

45 **Panos M. Pardalos and Pierre Hansen, Editors,** Data mining and mathematical programming, 2008

44 **Stanley Alama, Lia Bronsard, and Peter J. Sternberg, Editors,** Singularities in PDE and the calculus of variationa, 2007

43 **Andrew Granville, Melvyn B. Nathanson, and Jozsef Solymosi, Editors,** Additive combinatorics, 2007

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39 **P. Winternitz, D. Gomez-Ullate, A. Iserles, D. Levi, P. J. Olver, R. Quispel, and P. Tempesta, Editors,** Group theory and numerical analysis, 2005

38 **Jacques Hurtubise and Eyal Markman, Editors,** Algebraic structures and moduli spaces, 2004

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TITLES IN THIS SERIES

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The book is mostly devoted to the study of the prime factors of integers, their size and their quantity, to good bounds on the number of integers with different properties (for example, those with only large prime factors) and to the distribution of divisors of integers in a given interval. In particular, various estimates concerning smooth numbers are developed. A large emphasis is put on the study of additive and multiplicative functions as well as various arithmetic functions such as the partition function. More specific topics include the Erdős–Kac Theorem, cyclotomic polynomials, combinatorial methods, quadratic forms, zeta functions, Dirichlet series and L-functions. All these create an intimate understanding of the properties of integers and lead to fascinating and unexpected consequences. The volume includes contributions from leading participants in this active area of research, such as Kevin Ford, Carl Pomerance, Kannan Soundararajan and Gérald Tenenbaum.

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CRMP/46