Dynamics and Control of Mechanical Systems
The Falling Cat and Related Problems

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Editor
The Fields Institute
for Research in Mathematical Sciences

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Introduction

The first Fields Institute workshop, entitled "The Falling Cat and Related Problems", was held in early March 1992 in Waterloo, Ontario. The general theme was the application of methods from geometric mechanics and mathematical control theory to problems in the dynamics and control of freely rotating systems of coupled rigid bodies and related nonholonomic mechanical systems. The workshop was organized in an effort to bring together people with various backgrounds in the study of these systems and related issues.

Now is an exciting time to be working in this area, due to recent progress in geometric mechanics and in the theory of control and constrained optimal control problems on manifolds. The overlaps between control, geometry, and classical mechanics are strong, and are still in the process of being sorted out. Problems in the dynamics and control of freely rotating mechanical systems not only provide practical applications but also a natural setting in which to develop the theory.

Figure 1

The basic systems under consideration are *deformable* mechanical systems. Motions of such a system are determined by the usual quantities used to describe rigid body motions, namely rigid rotations of the entire system and translations of its mass center, as well as by changes in the shape of the system, for instance the relative orientations of a system of interconnected rigid bodies. These two components of motion are not independent of one another, and geometric notions such as holonomy play a central role in the analysis of motions of these systems.

Perhaps the most familiar example of such a system is the namesake of the volume, namely a cat, which, when dropped back–first from rest, is able to reorient itself and land on its feet in (at first glance) an apparent violation of the conservation of its angular momentum (see Figure 1). The fact that a cat can accomplish this by actively changing its shape without breaking any physical laws was fully accounted for with a straightforward computation in 1969 in a well–known paper of Kane and Scher (in fact there is considerable literature on this topic dating back to the turn of the century). This problem can be formulated naturally as a problem in the controllability of a suitable nonholonomic control system.

Another example of such a system is a dual–spin satellite, two examples of which are pictured in Figure 2. In this situation we have a mechanical system consisting of a rigid satellite (or ‘carrier’) connected to one or more rigid bodies (or ‘rotors’). The rotors can be constrained to rotate about axes fixed in the carrier as in Figure 2a, or can assume more general relative orientations; for instance, when connected to the carrier by a Cardan suspension as in Figure 2b, the rotor can assume arbitrary orientations relative to the outer body by appropriately adjusting the angles $\alpha$, $\beta$, and $\gamma$. In studying motions of this system, we might assume that its shape (given by the orientations of the rotors relative to the carrier) can be exactly controlled with internal torques (as above with the cat), or, alternatively, we might study the
free motions of the system without internal torques (as with, say, a ‘passive’ cat). Typical problems with this system might be:

(1) To determine the free dynamics of such a system;

(2) The construction of a controlled motion that transfers the carrier (or more generally the carrier and all of the rotors) between given orientations with a continuous, constant angular momentum motion; or

(3) The construction of a controlled, fixed–endpoint motion of the system which optimizes some other performance criterion (e.g., has minimal energy expenditure).

The first problem is the fundamental problem of classical mechanics, the second a problem in constructive controllability, and the third one of optimal control of a nonholonomic mechanical system. One can continue to give interesting examples of more involved problems, for instance if the carrier were a space telescope, we might want to construct a motion for which the carrier executes a complicated series of pointing manoeuvres.

Differential geometry provides not only a useful language for describing the dynamic and control–theoretic ideas involved in these problems but also a powerful set of tools that can be used in solving them. For instance, papers by Montgomery and by Shapere and Wilczek in the late 1980’s develop a gauge–theoretic interpretation of the motions of such a system. As an example, the motions of the system in Figure 2b evolve on the configuration space \( M = SO(3) \times SO(3) \), and if we assume the system has zero angular momentum and make appropriate choices of parameters, the motions of the system are those for which the bodies have opposite angular velocities. Using total kinetic energy as a metric, the tangent space at each point of \( M \) splits into the direct sum of vector spaces tangent to zero angular momentum motions and rigid motions of the whole system, i.e., the zero angular momentum constraint defines a horizontal distribution and consequently a connection on \( TM \). Assuming the action of the group is free, the quotient space \( S = M/G \), where \( G \) is the group of rigid rotations of the whole system, is a smooth manifold and \( M \) becomes a principal fibre bundle over \( S \) with fibre \( SO(3) \) over each point (if the motion is not free, singularities in shape space can result and the situation becomes especially interesting). A motion in \( S \) describes the relative orientation of the bodies (or shape of the system), and choosing a gauge (or reference point in the fibre over each point of \( S \)), the holonomy along a path in \( S \) determines a rigid rotation of the whole system that lifts a motion to \( M \). A similar description of motion is valid when angular momentum is nonzero. This construction, which effectively allows us to separate the rigid and shape components of the motion and study their interaction, is related to ideas from reduction of dynamical systems with symmetry, and the rigid component of motion induced by lifting a motion in \( S \) when reconstructing a motion in \( M \) is an example of a geometric phase. The practical applications of these ideas are many, ranging from the study of relative equilibria to the qualitative analysis and sometimes the explicit integration of the equations of motion for various problems.

At the same time, contemporary approaches to the (optimal) control of nonholonomically constrained control systems make a variety of naturally posed problems with these same systems accessible. Staying with the example of Figure 2b, it is
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easily seen that the distribution of vector fields tangent to zero angular momentum motions is not closed under the Lie bracket operation; in fact, the iterated Lie brackets of horizontal vector fields span $TM$, and Chow’s theorem implies that there is a zero angular momentum path connecting any two points of $M$, i.e., the system is controllable with zero angular momentum motions. When the system is assumed to have constant and nonzero angular momentum, more recent generalizations (e.g., [7]) of Chow’s theorem to motions on a Lie group in the presence of drift again imply controllability. An interesting optimal control problem of the form (3) above is that of minimizing the total integrated kinetic energy of the system over fixed–endpoint paths with the kinematic constraint of zero angular momentum (this is not the classical problem of finding free motions, in which only particular solutions have zero angular momentum; our present formulation requires internal torques between the bodies and is sometimes called a ‘vakanomic’ problem). The resulting problem is one of finding sub–Riemannian geodesics, (or an isoholonomic problem in Montgomery’s terminology). In particular, we want to minimize the length of a curve in the energy metric on $M$, but restricted to the bracket–generating distribution of vectors tangent to zero angular momentum motions. The restriction of the background metric to this distribution is called a sub–Riemannian metric. This problem, for the system of Figure 2b, can be completely integrated, and geometric methods have been used in the analysis of this problem for many specific variants of this system.

The use of geometric methods in understanding and solving problems of this sort remains a wide–open research area with diverse applications in fields from space dynamics to biomechanics. It is our sincere hope that it is just beginning.

The topic was well represented at the workshop, with papers presented on many aspects of the analysis of these systems, including: constructive controllability and explicit solution of optimal control problems for specific examples; analysis of relative equilibria; symmetry, reduction, and geometric phases; feedback stabilization of rigid body motions; symplectic integration methods; comparison and analysis of different formulations of the dynamics of general nonholonomic systems; applications of sub-Riemannian geometry, gauge theory, and exterior differential systems to constrained variational problems with these systems; the geometry of the rotation group, and specific applications to the design of space structures.

It is hoped that the present volume will be a useful addition to the volume “Dynamics and control of multibody systems” in the AMS series Contemporary Mathematics, Volume 97 (1989).

Bibliography


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The Falling Cat and Related Problems 
Michael J. Enos, Editor

This book contains a collection of papers presented at the Fields Institute workshop, "The Falling Cat and Related Problems", held in March 1992. The theme of the workshop was the application of methods from geometric mechanics and mathematical control theory to problems in the dynamics and control of freely rotating systems of coupled rigid bodies and related nonholonomic mechanical systems. This book will prove useful in providing insight into this new and exciting area of research.