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Normal Forms and Homoclinic Chaos

William F. Langford
Wayne Nagata
Editors

American Mathematical Society

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THE FIELDS INSTITUTE FOR RESEARCH IN MATHEMATICAL SCIENCES

Normal Forms and Homoclinic Chaos

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Providence, Rhode Island

The Fields Institute for Research in Mathematical Sciences

The Fields Institute is named in honour of the Canadian mathematician John Charles Fields (1863–1932). Fields was a remarkable man who received many honours for his scientific work, including election to the Royal Society of Canada in 1909 and to the Royal Society of London in 1913. Among other accomplishments in the service of the international mathematics community, Fields was responsible for establishing the world's most prestigious prize for mathematics research—the Fields Medal.

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Preface

This volume contains the proceedings of the Workshop on Normal Forms and Homoclinic Chaos, held at The Fields Institute for Research in Mathematical Sciences, November 13 to 16, 1992, in Waterloo, Canada, as a part of The Fields Institute Program Year on Dynamical Systems and Bifurcation Theory. The Workshop bridged the local and global analysis of dynamical systems, with emphasis on normal forms and the homoclinic phenomena which may arise in normal forms. Both dissipative and conservative, and both discrete and continuous systems were considered, including perturbed systems which cross these boundaries. Specific topics explored in the Workshop included: the effects of symmetry on normal forms and on the persistence of homoclinic orbits, exchange of stability, symmetry-breaking, resonances, integrability, local chaos, the effects of truncation of the normal form tail, efficient symbolic computation of normal forms, numerical computation of orbits in phase space, roundoff errors and shadowing. These topics were motivated by, and have application to, systems in physics, biology, engineering and other fields.

The Workshop assembled outstanding researchers as well as students from diverse backgrounds and geographical locations, to share ideas and to work together with longer term participants at the Institute. Some of the workshop visitors stayed at the Institute for periods preceding or following the Workshop, for extended discussions and collaborations. In all, there were 32 presentations at the Workshop, of which 14 are represented by papers in this volume.

The method of normal forms has its roots in the work of Poincaré [1892] and Birkhoff [1927]. It is a powerful technique for analyzing dynamical systems near nonhyperbolic equilibrium points, that is, points where the linearization has one or more eigenvalues with zero real part. This is the only place where local behaviour may become complicated (as follows from the Rectification Theorem and the Hartman-Grobman Theorem), and therefore "interesting" from both the mathematical and applied viewpoints. The method applies generally to parametrized families of differential equations on $\mathbb{R}^n$

$$\frac{dy}{dt} = f(y, \mu), \quad y \in \mathbb{R}^n, \ \ \alpha \in \mathbb{R}^p \tag{1}$$

where y is the state variable, μ is the parameter (both may be vectors) and t is an independent real variable usually referred to as "time". The parametrized vector field $f(y, \mu)$ is assumed to be smooth. In the construction of a normal form, a sequence of canonical changes of coordinates is performed, to successively "simplify" the Taylor polynomials, of increasing degree, in the series expansion of f in terms of y at the singular point. Classical work on normal forms often focussed on questions of convergence. In fact, convergence rarely holds for normal forms of practical interest. Modern work, as in this volume, deals with polynomial truncations of normal form expansions and their relationship to the flow generated by the original vector field.

The normal form for a given vector field is not unique, but it depends on the criteria chosen to simplify the Taylor polynomials. One particularly astute choice is always possible; one may eliminate all terms except those which commute with the flow generated by the adjoint of the linearization of the vector field at the equilibrium point (Belitskii [1981], Cushman and Sanders [1986], Elphick et al. [1987]). This choice has two advantages: it eliminates the nonuniqueness in the normal form in a consistent, rational manner, and, more importantly, it introduces an explicit equivariance (or symmetry) into the normal form which facilitates the analysis and may not have been apparent in the original system.

The method of normal forms gains power when coupled with the Center Manifold Theorem, which allows one to separate the (well-understood) hyperbolic behavior from the more interesting nonhyperbolic behavior in the neighborhood of an equilibrium. A major difficulty in the study of equation (1) is that its dimension may be large; it may even be infinite-dimensional, as for example in problems of fluid dynamics (Chossat and Iooss [1994]) or reaction-diffusion problems (Scott [1991]). However, often one is interested in changes in the qualitative behavior of (1). Such changes can occur locally only at values of μ for which (1) is nonhyperbolic; these are called bifurcation points. For example, the Hopf bifurcation occurs when a pair of complex-conjugate simple eigenvalues crosses the imaginary axis as μ varies; then, subject to certain generic transversality conditions, a periodic solution is created or destroyed around the equilibrium, see e.g. Golubitsky and Langford [1981]. Typically for (1), only a small number of eigenvalues cross the imaginary axis at one time; the corresponding eigenspaces (or modes) are finite dimensional. In this situation, the Center Manifold Theorem guarantees the existence of a manifold, called the center manifold, which is invariant for the flow of (1) and of the same dimension as, and tangent to, the sum of the (generalized) eigenspaces corresponding to those eigenvalues with zero real part. If the remaining eigenvalues have negative real part, then the flow off of this manifold is characterized by exponential decay towards the manifold. Thus, all the interesting behavior occurs on this low-dimensional center manifold, and it is desirable to reduce (1) to an equation for the flow on the center manifold. Fortunately for applications, there is an algorithm to calculate the center manifold and this reduced equation to any order, see Iooss and Adelmeyer [1992], Wiggins [1990]. The reduction to a center manifold has become so standard in recent years that most of the papers in this volume proceed from the assumption that it has been performed.

Let us now assume that (1) has been reduced to an equation on a center manifold. Then, in spite of the low dimensionality of (1), the analysis is still difficult, even if one is interested only in qualitative behavior. Because of the assumed nonhyperbolicity, the linear part no longer determines the local dynamics as it does in the hyperbolic case (Hartman-Grobman Theorem). The problem is intrinsically nonlinear and the best one can hope for is that the dynamics on the center manifold is determined by a finite degree truncation of the Taylor series, i.e. by a Taylor polynomial of (1). The number of terms in the Taylor polynomial grows rapidly with the degree, and quickly becomes unmanageable, even in dimension 2 or 3. The method of normal forms allows one to dramatically reduce the number of terms in these polynomials, without changing the qualitative behavior of solutions.

Often it is possible to give a complete description of the dynamics defined by this polynomial normal form (as in the case of classical Hopf bifurcation); several papers in this volume do precisely this. Of course, it still remains to show that the dynamics found for the truncated polynomial normal form persists when the higher order remainder or "tail" is restored. In some cases, this follows easily by transversality arguments (using the Implicit Function Theorem); in other cases the tail changes the qualitative dynamics and sometimes there remain open questions regarding the effects of the tail.

Symmetry is an important sub-theme of this volume. First, there is the normal form symmetry mentioned above, whereby the truncated normal form commutes with the flow generated by the adjoint of the linear part. It is this symmetry which characterizes the terms which remain in the normal form. In addition, the original system (1) may possess spatial or other symmetries, which can be expressed as equivariance with respect to the action of a group Γ, i.e.

$$f(\gamma y, \mu) = \gamma f(y, \mu) \tag{2}$$

for all $\gamma \in \Gamma$ and for all (y, μ). In this case, the reduction to the center manifold and further reduction to the normal form may be performed in such a way as to preserve the symmetry; more precisely, the normal form also satisfies the equivariance equation (2), where Γ is now understood to be an appropriate representation of the original symmetry group, on the center manifold. This additional symmetry has the effect of knocking out additional terms in the normal form and simplifying the analysis. Thus, symmetry is a double-edged sword: although symmetry can force multiple eigenvalues and increase the dimension of the center manifold, thus complicating the analysis, it also dramatically cuts down the number of terms in the normal form. The net effect is that new and rich structures with interesting dynamics, previously difficult to understand, become amenable to analysis. A classic example which provides a fascinating variety of bifurcations is the Taylor-Couette problem, see Golubitsky and Langford [1988] or Chossat and Iooss [1994]. Several new examples are presented in this volume.

The second main theme of the Workshop centered around homoclinic orbits; that is, solutions of (1) which have an equilibrium point (or limit cycle) as both α-limit and ω-limit sets. (The case where the α-limit and ω-limit sets are distinct points (or limit cycles) is properly called a heteroclinic orbit, but is included in this discussion without distinction.) Homoclinic orbits are not generic in (1); typically a homoclinic orbit will disappear under a small perturbation, for example due to a change in μ. Furthermore, they are generally difficult to calculate, or even to detect, due to their nonlocal nature. Yet, homoclinic orbits are of great interest, due in part to their important role in the onset of chaos, see Newhouse [1980], Wiggins [1990]. One context in which homoclinic orbits have been found to occur, and can even be calculated, is in the perturbations (or unfoldings) of a differential equation in normal form. Thus, paradoxically, the "global dynamics" of a homoclinic orbit can be found by "local analysis", in the neighborhood of a nonhyperbolic equilibrium point. In fact, to date, this has been the most effective means to study homoclinic behavior in real physical systems. Therefore, the linking of normal forms with homoclinic orbits and chaos in the theme of the Workshop and in this volume is seen to be natural and fruitful. Recently, it was observed that the addition of a third ingredient, namely symmetry, to this recipe can lead to a new phenomenon: normal forms of symmetric systems can have homoclinic orbits which are structurally stable; that is, which persist under small perturbations which respect the symmetry (see Field [1980], Armbruster et al. [1988], Campbell and Holmes [1992]). These structurally stable homoclinic orbits (or cycles of heteroclinic orbits) were subjects of discussion at the Workshop, and continue to be investigated intensively.

All that has been said above regarding the differential equation (1) and its flow, normal form and homoclinic orbits, holds equally well for *maps* and in particular diffeomorphisms; in fact chaos occurs even more readily for maps than for flows. Periodic orbits of (1) lead naturally to the definition of Poincaré maps, which are investigated in several papers of this volume. Similarly, the above comments apply also to systems which are Hamiltonian. The additional structure assumed by a Hamiltonian system at once facilitates the analysis (e.g. by providing first integrals) and leads to complex dynamics, as seen in Marsden [1992] as well as this volume.

The Workshop represented by this volume was part of the Program Year on Dynamical Systems and Bifurcation Theory, held at The Fields Institute for Research in Mathematical Sciences, from September 1992 to August 1993. The Program Committee consisted of J. Chadam, L. Glass, W. Langford, J. Marsden and W. Shadwick. The main areas of emphasis were: bifurcations with symmetry; low dimensional dynamics including normal forms and invariant manifolds; geometric methods from Hamiltonian mechanics including higher dimensional Melnikov methods and Lagrangian reduction; applications to many disciplines, especially to problems involving pattern formation in partial differential equations. There were seven workshops held during the Program Year, including this one; proceedings of all the workshops are being published.

The Fields Institute for Research in Mathematical Sciences is supported by the Ontario Ministry of Education and Training and by the Natural Sciences and Engineering Research Council of Canada, and also by the sponsoring and affiliated universities. The Fields Institute has three main mandates: to promote leading edge research in the mathematical sciences, to increase opportunities for graduate and postdoctoral training in Ontario and Canada and to foster interactions between university based research and users of mathematics in industry. In addition, the Institute serves as a resource for mathematics education at all levels.

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William F. Langford and Wayne Nagata, Editors

This volume presents new research on normal forms, symmetry, homoclinic cycles, and chaos, from the Workshop on Normal Forms and Homoclinic Chaos held during The Fields Institute Program Year on Dynamical Systems and Bifurcation Theory in November 1992, in Waterloo, Canada. The workshop bridged the local and global analysis of dynamical systems with emphasis on normal forms and the recently discovered homoclinic cycles which may arise in normal forms.

Specific topics covered in this volume include...

- normal forms for dissipative, conservative, and reversible vector fields, and for symplectic maps;
- the effects of symmetry on normal forms;
- the persistence of homoclinic cycles;
- symmetry-breaking, both spontaneous and induced;
- mode interactions;
- resonances;
- intermittency;
- numerical computation of orbits in phase space;
- applications to flow-induced vibrations and to mechanical and structural systems;
- general methods for calculation of normal forms;
- chaotic dynamics arising from normal forms.

Of the 32 presentations given at this workshop, 14 of them are represented by papers in this volume.

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