Perspectives on Noncommutative Geometry

Masoud Khalkhali
Guoliang Yu
Editors
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The Fields Institute for Research in Mathematical Sciences

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2000 Mathematics Subject Classification. Primary 58B34; Secondary 19D55, 16T05, 18G30.

Library of Congress Cataloging-in-Publication Data
Perspectives on noncommutative geometry / Masoud Khalkhali, Guoliang Yu, editors.
   p. cm. — (Fields Institute Communications)
   Includes bibliographical references.
   512.55—dc23
   2011032554

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This publication was prepared by the Fields Institute.

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Preface

During May 27–31, 2008, the Noncommutative Geometry Workshop was held at the Fields Institute as part of the thematic program on operator algebras. This was the second major conference on the subject organized at the Fields Institute, the first being in 1995. It was interesting to appreciate how much the subject has progressed between these two events. The present volume consists primarily of articles by speakers in this workshop.

Roughly speaking, noncommutative geometry concerns itself with the study of noncommutative spaces. These ‘spaces’ are usually represented by a noncommutative algebra that replaces the coordinate algebra in the commutative case. Examples include highly singular spaces such as the space of leaves of a foliation, the unitary dual of a noncompact group, and more generally, ‘bad quotients’ of classical spaces.

Initiated and pioneered by Alain Connes since 1980, in its initial stage noncommutative geometry was mostly inspired by global analysis, topology, operator algebras, and quantum physics, as they show up in areas such as index theory, foliation theory, and quantum statistical mechanics. Its main applications were to settle some long standing conjectures such as the Novikov conjecture, and the Baum-Connes conjecture in topology and analysis. The main tools here are cyclic cohomology, K-theory and K-homology, and KK-theory. Next came the impact of spectral geometry and the way the spectrum of a geometric operator like the Laplacian informs us about the geometry and topology of a manifold, as in the celebrated Weyl’s law. This is now subsumed and vastly generalized through Connes’ notion of spectral triples, which is a centerpiece of noncommutative Riemannian geometry and applications of noncommutative geometry to particle physics. Finally, in recent years we have witnessed the impact of number theory, algebraic geometry, and the theory of motives and quantum field theory on noncommutative geometry, and a strong interaction between these areas is gradually emerging. All these aspects of the field were reflected and touched upon in lectures by the invited speakers at the workshop.

During the workshop, Alain Connes delivered his Fields Institute Distinguished Lectures Series. He gave a series of three lectures on the frontiers of research in the subject, with titles The spectral characterization of manifolds, A CKM invariant in Riemannian geometry, and About the field with one element. The reader can refer to the web page\(^1\) of the Fields Institute for abstracts of these lectures.

We would like to thank Carl Riehm and Debbie Iscoe of Fields Institute Publications for their patience and professional assistance. We also thank all the contributors for their contributions. It is a pleasure to thank George Elliott for supporting the original idea of a workshop on noncommutative geometry during the Fields Institute thematic program on operator algebras. The workshop was financially

\(^1\)http://www.fields.utoronto.ca/programs/scientific/07-08/noncommutative/
supported by the Fields Institute and the NSF, and we would like to thank both institutions for their support. It is a pleasure to thank Arthur Greenspoon for checking the entire manuscript and suggesting many improvements. Finally we would like to warmly thank Matilde Marcolli who coorganized this workshop with us.

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This volume represents the proceedings of the Noncommutative Geometry Workshop that was held as part of the thematic program on operator algebras at the Fields Institute in May 2008.

Pioneered by Alain Connes starting in the late 1970s, noncommutative geometry was originally inspired by global analysis, topology, operator algebras, and quantum physics. Its main applications were to settle some long-standing conjectures, such as the Novikov conjecture and the Baum-Connes conjecture.

Next came the impact of spectral geometry and the way the spectrum of a geometric operator, like the Laplacian, holds information about the geometry and topology of a manifold, as in the celebrated Weyl law. This has now been vastly generalized through Connes’ notion of spectral triples.

Finally, recent years have witnessed the impact of number theory, algebraic geometry and the theory of motives, and quantum field theory on noncommutative geometry. Almost all of these aspects are touched upon with new results in the papers of this volume.

This book is intended for graduate students and researchers in both mathematics and theoretical physics who are interested in noncommutative geometry and its applications.