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Global Dynamics, Phase Space Transport, Orbits Homoclinic to Resonances, and Applications

Stephen Wiggins

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The Fields Institute for Research in Mathematical Sciences

The Fields Institute is named in honour of the Canadian mathematician John Charles Fields (1863–1932). Fields was a remarkable man who received many honours for his scientific work, including election to the Royal Society of Canada in 1909 and to the Royal Society of London in 1913. Among other accomplishments in the service of the international mathematics community, Fields was responsible for establishing the world's most prestigious prize for mathematics research—the Fields Medal.

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Dedication

This book is dedicated to Pat Sethna, in honor of his many contributions to applied nonlinear dynamics.

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Preface

This monograph consists of a series of weekly lectures that I gave at the Fields Institute during January through March of 1993. All of the material presented here represents joint work with colleagues Tony Leonard, Dave McLaughlin, Ed Overman II, and C. Xiong, and former Caltech students Darin Beigie, Roberto Camassa, György Haller, Tasso Kaper, Gregor Kovačič, and Vered Rom-Kedar.

I would like to thank the organizers of the Dynamical Systems and Bifurcation Theory year at the Fields Institute, John Chadam, Leon Glass, Bill Langford, Jerry Marsden, and Bill Shadwick, for making it possible for me to spend time at the Fields Institute and to give these lectures. I would like to thank Wayne Nagata and Dan Rusu for providing an early draft of the lectures. I would like to also acknowledge the wonderful help and support from the Fields Institute staff, Sheri Albers, Ron Hosler, Judy Motts, and Jaqua Taylor. Their efforts, in too many areas to mention, made my visit both very pleasant and productive. My thanks to Willem Sluis and Robin Skinner of the Fields Institute for their proofreading. I am very grateful to Brenda Law of the University of Waterloo and Liz Reidt for taking my rough LaTeX files and transforming them into a beautiful, typeset quality form. Also, my thanks to Sue Embro and Sandra Valeriote of the Fields Institute for their coordinating efforts towards the final production of this document. I would like to thank my graphic artist at Caltech, Cecelia Lin, for her work on the many figures under a very tight deadline. None of these lectures would have been given without the support of my colleague, in Applied Mechanics at Caltech, Jim Knowles. Jim took over my teaching load, on top of his own, so that I could spend three months at the Fields Institute.

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Steve Wiggins

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**Global Dynamics, Phase Space Transport,
Orbits Homoclinic to Resonances, and Applications**

Stephen Wiggins

This monograph, a series of lectures delivered by Stephen Wiggins at the Fields Institute in early 1993, is concerned with the geometrical viewpoint of the global dynamics of nonlinear dynamical systems. With appropriate examples and concise explanations, Wiggins unites many different topics into one volume and makes a unique contribution to the field. Engineers, physicists, chemists, and mathematicians who work on issues related to the global dynamics of nonlinear dynamical systems will find these lectures very useful.

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