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Contents

Preface

This book contains recent results about the global dynamics defined by a class of delay differential equations which model basic feedback mechanisms and arise in a variety of applications such as neural networks. We describe in detail the geometric structure of a fundamental invariant set, which in special cases is the global attractor, and the asymptotic behaviour of solution curves on it.

Our approach makes use of advanced tools which in recent years have been developed for the investigation of infinite-dimensional dynamical systems: Local invariant manifolds and inclination lemma for noninvertible maps, Floquet theory for delay differential equations, a-priori estimates controlling the growth and decay of solutions with prescribed oscillation frequency, a discrete Lyapunov functional counting zeros, methods to represent invariant sets as graphs, Poincaré–Bendixson techniques for classes of delay differential systems.

Several appendices provide the general results needed in our case study, so that the presentation is self-contained. Some of these general results seem not to be available elsewhere in the literature. We mention in particular Appendix II on smooth infinite-dimensional center-stable manifolds for maps. We believe that the results in the appendices will be useful also for future studies of more complicated attractors of delay and partial differential equations.

A brief description of a part of our main result is that for the delay differential equation $\dot{x}(t) = -\mu x(t) + f(x(t-1))$ with $\mu > 0$ and increasing bounded C^1-function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$, under natural and mild additional conditions, the leading 3-dimensional local unstable manifold at the stationary point 0 extends in forward time to a smooth solid spindle with singularities at its tips, which are further stationary points, both stable and attractive; an invariant smooth disk of solution curves winding from 0 towards a bordering unstable periodic orbit splits the spindle into invariant halves each of which is attracted to one of its tips.

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T. Krisztin, H. O. Walther and J. Wu
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Index

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