



# FIELDS INSTITUTE MONOGRAPHS

THE FIELDS INSTITUTE FOR RESEARCH IN MATHEMATICAL SCIENCES

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Tibor Krisztin  
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## The Fields Institute for Research in Mathematical Sciences

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The Fields Institute for Research in Mathematical Sciences is supported by grants from the Ontario Ministry of Education and Training and the Natural Sciences and Engineering Research Council of Canada. The Institute is sponsored by McMaster University, the University of Toronto, the University of Waterloo, and York University and has affiliated universities in Ontario and across Canada.

1991 *Mathematics Subject Classification*. Primary 34K15;  
Secondary 58F12, 58F22, 34C30.

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### Library of Congress Cataloging-in-Publication Data

Krisztin, Tibor, 1956–

Shape, smoothness, and invariant stratification of an attracting set for delayed monotone positive feedback / Tibor Krisztin, Hans-Otto Walther, Jianhong Wu.

p. cm. — (Fields institute monographs, ISSN 1069-5273 ; 11)

Includes bibliographical references and index.

ISBN 0-8218-1074-X (alk. paper)

1. Delay differential equations—Numerical solutions. 2. Initial value problems. 3. Attractors (Mathematics) I. Walther, Hans-Otto. II. Wu, Jianhong. III. Title. IV. Series.

QA371.K735 1999

515'.35—dc21

98-44070

CIP

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10 9 8 7 6 5 4 3 2 1 04 03 02 01 00 99

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## Preface

This book contains recent results about the global dynamics defined by a class of delay differential equations which model basic feedback mechanisms and arise in a variety of applications such as neural networks. We describe in detail the geometric structure of a fundamental invariant set, which in special cases is the global attractor, and the asymptotic behaviour of solution curves on it.

Our approach makes use of advanced tools which in recent years have been developed for the investigation of infinite-dimensional dynamical systems: Local invariant manifolds and inclination lemma for noninvertible maps, Floquet theory for delay differential equations, a-priori estimates controlling the growth and decay of solutions with prescribed oscillation frequency, a discrete Lyapunov functional counting zeros, methods to represent invariant sets as graphs, Poincaré–Bendixson techniques for classes of delay differential systems.

Several appendices provide the general results needed in our case study, so that the presentation is self-contained. Some of these general results seem not to be available elsewhere in the literature. We mention in particular Appendix II on smooth infinite-dimensional center-stable manifolds for maps. We believe that the results in the appendices will be useful also for future studies of more complicated attractors of delay and partial differential equations.

A brief description of a part of our main result is that for the delay differential equation  $\dot{x}(t) = -\mu x(t) + f(x(t-1))$  with  $\mu > 0$  and increasing bounded  $C^1$ -function  $f : \mathbb{R} \rightarrow \mathbb{R}$  with  $f(0) = 0$ , under natural and mild additional conditions, the leading 3-dimensional local unstable manifold at the stationary point 0 extends in forward time to a smooth solid spindle with singularities at its tips, which are further stationary points, both stable and attractive; an invariant smooth disk of solution curves winding from 0 towards a bordering unstable periodic orbit splits the spindle into invariant halves each of which is attracted to one of its tips.

The major part of this work was carried out while T. Krisztin and J. Wu were supported by the Alexander von Humboldt Foundation during their guest stay with Humboldt fellowships at the University of Giessen in 1996–97. Both are indebted to the Foundation for its generous support and to the Institute of Mathematics at the University of Giessen for the kind hospitality during their visit. We are grateful to Professor J. Kincses for calling our attention to some topological results used in Chapter 16.

We would like to acknowledge the support from the Hungarian National Foundation for Scientific Research (T. Krisztin), and from the Natural Sciences and Engineering Research Council of Canada (J. Wu).

T. Krisztin, H. O. Walther and J. Wu  
June 1998

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## Bibliography

- [1] Abraham, R. and Robbin, J., *Transversal Mappings and Flows*, Benjamin, New York, 1967.
- [2] Ammar, Y., *Eine dreidimensionale invariante Mannigfaltigkeit für autonome Differentialgleichungen mit Verzögerung*, Dissertation, Universität München, 1993.
- [3] Arino, O., *A note on "The Discrete Lyapunov Function ..."*, J. Differential Equations **104** (1993), 169–181.
- [4] Arino, O. and Krisztin, T., *The 2-dimensional attractor of a differential equation with state-dependent delay*, in preparation.
- [5] Arino, O. and Séguier, P., *Existence of oscillating solutions for certain differential equations with delay*, Functional Differential Equations and Approximation of Fixed Points (H.-O. Peitgen, H.-O. Walther, eds.), Lecture Notes in Math., Vol. 730, pp. 46–64, Springer-Verlag, New York, 1979.
- [6] Arino, O. and Séguier, P., *Contribution à l'étude des comportements des solutions d'équation différentielle à retard par des méthodes de monotonie et de bifurcation*, Thèses d'Etat, Bordeaux I, Pau, 1980.
- [7] Arndt, M., *Repräsentation räumlicher und zeitlicher Stetigkeit durch Synchronisation neuronaler Signale*, Dissertation, Universität Marburg, Fachbereich Physik, 1983.
- [8] Bélair, J., *Stability in a model of a delayed neural network*, J. Dynamics and Differential Equations **5** (1993), 607–623.
- [9] Bélair, J., Campbell, S. A. and van den Driessche, P., *Frustration, stability, and delay-induced oscillations in a neural network model*, SIAM J. Appl. Math. **56** (1996), 245–255.
- [10] Bélair, J. and Dufour, S., *Stability in a three-dimensional system of delay-differential equations*, Canad. Appl. Math. Quarterly **4** (1996), 135–156.
- [11] Bellman, R. and Cooke, K. L., *Differential-Difference Equations*, Academic Press, New York, 1963.
- [12] Bing, R. H., *The Geometric Topology of 3-Manifolds*, Amer. Math. Soc. Colloq. Publ., Vol. 40, Providence, RI, 1983.
- [13] Braun, H., *Personal communication*.
- [14] Bredon, G. E., *Topology and Geometry*, Springer-Verlag, New York, 1995.
- [15] Burton, T. A. and Hatvani, L., *Stability theorems for nonautonomous functional differential equations by Lyapunov functionals*, Tôhoku Math. J. **41** (1989), 65–104.
- [16] Cao, Y., *The discrete Lyapunov function for scalar delay differential equations*, J. Differential Equations **87** (1990), 365–390.
- [17] Chen, X.-Y., Hale, J. K. and Tan, B., *Invariant foliations for  $C^1$  semigroups in Banach spaces*, J. Differential Equations **139** (1997), 283–318.
- [18] Chow, S.-N., Diekmann, O. and Mallet-Paret, J., *Stability, multiplicity and global continuation of symmetric periodic solutions of a nonlinear Volterra integral equation*, Japan J. Appl. Math. **2** (1985), 433–469.
- [19] Chow, S.-N. and Walther, H.-O., *Characteristic multipliers and stability of symmetric periodic solutions of  $\dot{x}(t) = g(x(t-1))$* , Trans. Amer. Math. Soc. **307** (1988), 127–142.
- [20] Cohen, M. A. and Grossberg, S., *Absolute stability of global pattern formation and parallel memory storage by competitive neural networks*, IEEE Trans. SMC. **13** (1983), 815–826.
- [21] Deimling, K., *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.
- [22] Diekmann, O., van Gils, S. A., Verduyn Lunel, S. M. and Walther, H.-O., *Delay Equations, Functional-, Complex-, and Nonlinear Analysis*, Springer-Verlag, New York, 1995.
- [23] Dieudonné, J., *Foundations of Modern Analysis*, Academic Press, New York, 1960.

- [24] van den Driessche, P. and Zou, X. F., *Global attractivity in delayed Hopfield neural network models*, SIAM J. Math. Anal., to appear.
- [25] Fiedler, F. and Mallet-Paret, J., *Connections between Morse sets for delay differential equations*, J. reine angew. Math. **397** (1989), 23–41.
- [26] Györi, I., *Connections between compartmental systems with pipes and integrodifferential equations*, Mathematical Modelling **7** (1987), 1215–1238.
- [27] Hale, J. K., *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.
- [28] Hale, J. K., *Asymptotic Behavior of Dissipative Systems*, Amer. Math. Soc., Providence, RI, 1988.
- [29] Hale, J. K. and Lin, X. B., *Symbolic dynamics and nonlinear semiflows*, Annali Mat. Pura Appl. **144** (1986), 693–709.
- [30] Hale, J. K. and Verduyn Lunel, S. M., *Introduction to Functional Differential Equations*, Springer-Verlag, New York, 1993.
- [31] Hatvani, L. and Krisztin, T., *Asymptotic stability for a differential-difference equation containing terms with and without delay*, Acta Sci. Math. (Szeged) **60** (1995), 371–384.
- [32] Herz, A. V. M., *Global Analysis of recurrent neural networks*, in Models of Neural Networks, Vol. 3 (E. Domany, J. L. van Hemmen and K. Schulten, eds.), Springer-Verlag, New York, 1994.
- [33] Hirsch, M., *Stability and convergence in strongly monotone dynamical systems*, J. reine angew. Math. **383** (1988), 1–53.
- [34] Hocking, J. G. and Young, G. S., *Topology*, Addison-Wesley, Dover, 1988.
- [35] Hopfield, J. J., *Neural networks and physical systems with emergent collective computational abilities*, Proc. Natl. Acad. Sci. **79** (1982), 2554–2558.
- [36] Hopfield, J. J., *Neurons with graded response have collective computational properties like two-stage neurons*, Proc. Natl. Acad. Sci. **81** (1984), 3088–3092.
- [37] Iooss, G., *Bifurcation of Maps and Applications*, North-Holland, Amsterdam, 1979.
- [38] Krisztin, T., *On the convergence of solutions of functional differential equations*, Acta Sci. Math. **43** (1981), 45–54.
- [39] Krisztin, T., *Convergence of solutions of a nonlinear integrodifferential equation arising in compartmental systems*, Acta Sci. Math. **47** (1984), 471–485.
- [40] Krisztin, T., *An invariance principle of Lyapunov–Razumikhin type and compartmental systems*, in Proc. of the First World Congress of Nonlinear Analysts (V. Lakshmikantham, ed.), pp. 1371–1379, Walther de Gruyter, Berlin, 1996.
- [41] Lani-Wayda, B. and Walther, H.-O., *Chaotic motion generated by delayed negative feedback, Part I: a transversality criterion*, Differential and Integral Equations **8** (1995), 1407–1452.
- [42] Mallet-Paret, J., *Morse decompositions for differential delay equations*, J. Differential Equations **72** (1988), 270–315.
- [43] Mallet-Paret, J. and Sell, G., *Systems of differential delay equations: Floquet multipliers and discrete Lyapunov functions*, J. Differential Equations **125** (1996), 385–440.
- [44] Mallet-Paret, J. and Sell, G., *The Poincaré–Bendixon theorem for monotone cyclic feedback systems with delay*, J. Differential Equations **125** (1996), 441–489.
- [45] Mallet-Paret, J. and Walther, H.-O., *Rapid oscillations are rare in scalar systems governed by monotone negative feedback with a time delay*, Math. Inst., University of Giessen, 1994, preprint.
- [46] Marcus, C. M. and Westervelt, R. M., *Stability of analog neural networks with delay*, Phys. Rev. A **39** (1989), 347–356.
- [47] Morita, M., *Associative memory with non-monotone dynamics*, Neural Networks **6** (1993), 115–126.
- [48] Neugebauer, A., *Invariante Mannigfaltigkeiten und Neigungslemmata für Abbildungen in Banachräumen*, Diploma thesis, Universität München, 1988.
- [49] Olien, L. and Bélair, J., *Bifurcations, stability, and monotonicity properties of a delayed neural network model*, Physica D. **102** (1997), 349–363.
- [50] Pakdaman, K., Malta, C. P., Grotta-Ragazzo, C. and Vibert, J.-F., *Effect of delay on the boundary of the basin of attraction in a self-excited single neuron*, Neural Computation **9** (1997), 319–336.
- [51] Pakdaman, K., Grotta-Ragazzo, C., Malta, C. P., Arino, O. and Vibert, J.-F., *Effect of delay on the boundary of the basin of attraction in a system of two neurons*, preprint.
- [52] Rinow, W., *Lehrbuch der Topologie*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1975.

- [53] Schoenflies, A., *Die Entwicklung der Lehre von den Punktmannigfaltigkeiten. Bericht erstattet der Deutschen Mathematiker-Vereinigung, Teil II*, J.-Ber. Deutsch. Math.-Verein., Ergänzungsband II, 1908.
- [54] Smith, H. L., *Monotone semiflows generated by functional differential equations*, J. Differential Equations **66** (1987), 420–442.
- [55] Smith, H. L., *Oscillation and multiple steady states in a cyclic gene model with repression*, J. Math. Biol. **25** (1987), 169–190.
- [56] Smith, H. L., *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems*, Amer. Math. Soc., Providence, RI, 1995.
- [57] Smith, H. L. and Thieme, H. R., *Strongly order preserving semiflows generated by functional differential equations*, J. Differential Equations **93** (1991), 332–363.
- [58] Tank, D. and Hopfield, J. J., *Simple neural optimization networks: an A/D converter, single decision circuit and a linear programming circuit*, IEEE Trans. Circ. Syst. **33** 1986, 533–541.
- [59] Vanderbauwhede, A. and van Gils, S. A., *Center manifolds and contractions on a scale of Banach spaces*, J. Functional Analysis **71** (1987), 209–224.
- [60] Vanderbauwhede, A. and Iooss, G., *Center manifold theory in infinite dimensions*, in Dynamics Reported (New Series), Vol. 1 (C.K.R.T. Jones, U. Kirchgraber and H.-O. Walther, eds.), Springer-Verlag, New York, 1992.
- [61] Walther, H.-O., *Über Ejektivität und periodische Lösungen bei Funktionaldifferentialgleichungen mit verteilter Verzögerung*, Habilitationsschrift, Universität München, 1977.
- [62] Walther, H.-O., *On instability,  $\omega$ -limit sets and periodic solutions of nonlinear autonomous differential delay equations*, in *Functional Differential Equations and Approximation of Fixed Points* (H.-O. Peitgen, H.-O. Walther eds.), Lecture Notes in Math., Vol. 730, pp. 489–503, Springer-Verlag, New York, 1979.
- [63] Walther, H.-O., *Homoclinic solutions and chaos in  $\dot{x}(t) = f(x(t-1))$* , Nonlinear Anal. **5** (1981), 775–788.
- [64] Walther, H.-O., *Bifurcation from periodic solutions in functional differential equations*, Math. Z. **182** (1983), 269–289.
- [65] Walther, H.-O., *Hyperbolic periodic solutions, heteroclinic connections and transversal homoclinic points in autonomous differential delay equations*, Memoirs of the Amer. Math. Soc., Vol. 402, Amer. Math. Soc., Providence, RI, 1989.
- [66] Walther, H.-O., *A differential delay equation with a planar attractor*, in *Proc. of the Int. Conf. on Differential Equations*, Université Cadi Ayyad, Marrakech, 1991.
- [67] Walther, H.-O., *An invariant manifold of slowly oscillating solutions for  $x'(t) = -\mu x(t) + f(x(t-1))$* , J. reine angew. Math. **414** (1991), 67–112.
- [68] Walther, H.-O., *On Floquet multipliers of periodic solutions of delay equations with monotone nonlinearities*, in *Int. Symp. Functional Differential Equations (Kyoto, 1990)*, pp. 349–356, World Scientific, Singapore, 1991.
- [69] Walther, H.-O., *Unstable manifolds of periodic orbits of a differential delay equation*, Contemp. Math. **129** (1992), 177–239.
- [70] Walther, H.-O., *The 2-dimensional attractor of  $\dot{x}(t) = -\mu x(t) + f(x(t-1))$* , Memoirs of the Amer. Math. Soc., Vol. 544, Amer. Math. Soc., Providence, RI, 1995.
- [71] Walther, H.-O. and Yebdri, M., *Smoothness of the attractor of almost all solutions of a delay differential equation*, Dissertationes Mathematicae **368**, 1997.
- [72] Wu, J., *Symmetric functional differential equations and neural networks with memory*, Trans. Amer. Math. Soc., to appear.

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ISBN 0-8218-1074-X



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