Meromorphic Functions and Linear Algebra

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Olavi Nevanlinna
The Fields Institute
for Research in Mathematical Sciences

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# CONTENTS

**PREFACE**

ix

**PROLOGUE**

1

Tapping away in an evening at Djursholm 1
What does an epsilon weigh? 1
Red wine at the Stock Exchange Club 2
Ice-cream in Madison 2
Exact equality 3
The deficiency of values 5
Zurich 5
Beautiful to look at, but... 6
The unbearable ease of using norms 6
Centenary Colloquium in Joensuu 7
Two basic tasks, stability first 7
And then accelerating the iteration 9
Factoring the resolvent 10
In the Hermann Weyl lecture hall 11
A quiet life in Warsaw 12
Finally, in Kirkkonummi 12

**FIRST CHAPTER**

15

Resolvent 15
Cauchy-integral 20

**SECOND CHAPTER**

23

Entire functions 23
Taylor coefficients 24
Meromorphic functions 25
The first main theorem 30
Cartan’s identity 31
Order and type for meromorphic functions 32
Boutroux-Cartan lemma 33
Bound along a circle 34
Representation theorems 36

**THIRD CHAPTER**

37

Analytic vector valued functions 37
Subharmonic functions 37
Meromorphic vector valued functions 38
Rational functions
When is the inverse also meromorphic
A simple estimate for matrices

FOURTH CHAPTER
A product form for matrices
Singular value decomposition
Basic inequalities for singular values and eigenvalues
The total logarithmic size of a matrix
Some basic properties of the total logarithmic size
Direct sum, Kronecker product and Hadamard product

FIFTH CHAPTER
The total logarithmic size is subharmonic
Behavior near poles
Introducing $T_1$ for matrix valued functions
Basic identity for inversion
Extension to trace class
How to work outside the trace class

SIXTH CHAPTER
Perturbation results
Special results for resolvents
Powers and their resolvents
Bounded characteristics
What if small perturbation means small in norm

SEVENTH CHAPTER
Combining a scalar function with an operator
Representing $F$ as $G/g$
Representations for the resolvent
Decay of spectral polynomials
Robust bounds for Krylov solvers
A bound for spectral projectors

EIGHTH CHAPTER
Approximate polynomial degree of an analytic function
Some properties of the approximate polynomial degree
Approximate rational degree of a meromorphic function
Spijker’s lemma
Power bounded operators and bounds for the Laurent coefficients

NINTH CHAPTER
Growth of associated scalar functions
Locally algebraic and locally almost algebraic operators

TENTH CHAPTER
Exceptional values
Simple asymptotics for resolvents of matrices
Eigenvalues and exceptional values
Deficient operators
CONTENTS

EPILOGUE
Lecturing and typing in Toronto 133
Fishing and finishing in Karjalohja 133

BIBLIOGRAPHY 135
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PREFACE

This monograph is based on lectures which were given in two phases. In the fall of 1995 I gave a series of lectures at the Helsinki University of Technology and in October 2001 at the Fields Institute in Toronto.

With this monograph I hope to demonstrate that viewing the resolvent of a matrix as a meromorphic function rather than just analytic outside the eigenvalues gives a lot of new insight. In low rank perturbations the eigenvalues - and pseudospectra - may move dramatically but underneath there is still much which is preserved. Since this has practical implications e.g. to preconditioning, I am trying to present the ideas in a simple and self contained form, accessible for the researchers in the numerical linear algebra community. However, some of the results are more natural to set up in infinite dimensional spaces as the asymptotics is then richer.

The monograph is organized as follows. In the first chapter the resolvent is explicitly written down. The second chapter gives a summary of elementary value distribution theory - without going into the second main theorem. The third chapter then discusses vector valued analytic and meromorphic functions. The main new “tool”, the total logarithmic size of a matrix is introduced in chapter four. It is a nonlinear tool for linear algebra and it allows one to generalize the first main theorem from the scalar valued case for matrix valued functions. This is done in chapter five. In chapter six we discuss some applications and show in particular that the growth of the resolvent as a meromorphic function is robust under low rank perturbations. The seventh chapter discusses first operators of the form

\[ z \mapsto f(zA) \]

where \( f(z) \) is a scalar meromorphic function and \( A \) a bounded operator such that its resolvent is a meromorphic function. Another topic discussed is bounds for Krylov methods for solving

\[ x = Ax + b. \]

We connect the decay of the bounds for the growth of the resolvent as a meromorphic function and as this is robust in low rank perturbations so are our bounds. Chapter eight gives a new tool into approximation theory. The growth of a meromorphic function is studied by approximating it by rational functions. The results are then applied to Kreiss matrix theorem, power boundedness and other related questions. In the ninth chapter we associate with a given operator valued meromorphic function \( F \) scalar functions

\[ f_{x,y^*} : z \mapsto y^*(F(z)x), \]

and ask whether there are unit vectors \( x, y^* \) such that the growth of \( F \) as a meromorphic function can be seen from the growth of \( f_{x,y^*} \). The last chapter gives a
link between the defects in value distribution theory and defective eigenvalues of a matrix.

In addition I have included an epilogue and a prologue to explain how I got the ideas in the first place.

I can be reached via e-mail at Olavi.Nevanlinna@hut.fi. Some software is available at URL http://www.math.hut.fi/annex/.

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Kirkkonummi, Finland
September 10, 2002
BIBLIOGRAPHY


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The main goal of the book is to study the behavior of the resolvent of a matrix under the perturbation by low rank matrices. Whereas the eigenvalues, that is, the poles of the resolvent, and the pseudospectra, that is, the sets where the resolvent takes large values, can move dramatically under such perturbations, the growth of the resolvent as a matrix-valued meromorphic function remains essentially unchanged. This has practical implications to the analysis of iterative solvers for large systems of linear algebraic equations.

The book first gives an introduction to the basics of value distribution theory of meromorphic scalar functions. Then it introduces a new nonlinear tool for linear algebra, the total logarithmic size of a matrix, which allows for a nontrivial generalization of Rolf Nevanlinna's characteristic function from the scalar theory to matrix- and operator-valued functions. In particular, the theory of perturbations by low rank matrices becomes possible. As an example, if the spectrum of a normal matrix collapses under a low rank perturbation, there is always a compensation in terms of the loss of orthogonality of the eigenvectors. This qualitative phenomenon is made quantitative by using the new tools. Applications are given to rational approximation, to the Kreiss matrix theorem, and to convergence of Krylov solvers.

Some results appear here for the first time, while the rest are extended from recent papers of the author. The book is intended for researchers in mathematics in general and especially for those working in numerical linear algebra. Much of the book is understandable if the reader has a good background in linear algebra and a first course in complex analysis.