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Foreword

In the study of a mathematical system, algebraic structures allow for the discovery of more information. This is the motor behind the success of many areas of mathematics such as algebraic geometry, algebraic combinatorics, algebraic topology and others. This was certainly the motivation behind the observation of G.-C. Rota stating that various combinatorial objects possess natural product and coproduct structures. These structures give rise to a graded Hopf algebra, which is usually referred to as a combinatorial Hopf algebra. Typically, it is a graded vector space where the homogeneous components are spanned by finite sets of combinatorial objects of a given type and the algebraic structures are given by some constructions on those objects.

Recent foundational work has constructed many interesting combinatorial Hopf algebras and uncovered new connections between diverse subjects such as combinatorics, algebra, geometry, and theoretical physics. This has expanded the new and vibrant subject of combinatorial Hopf algebras. To give a few instances:

- Connes and Kreimer showed that a certain renormalization problem in quantum field theory can be encoded and solved using a Hopf algebra spanned by rooted trees.
- Loday and Ronco showed that a Hopf algebra based on planar binary trees is the free dendriform algebra on one generator. This is true for many types of algebras; the free algebra on one generator is a combinatorial Hopf algebra.
- In the context of polytope theory, some interesting enumerative combinatorial invariants induce a Hopf morphism from a Hopf algebra of posets to the Hopf algebra of quasi-symmetric functions.
- Krob and Thibon showed that the representation theory of the Hecke algebras at $q = 0$ is intimately related to the Hopf algebra structure of quasi-symmetric functions and non-commutative symmetric functions.

Some of the latest research in these areas has been the subject of a series of recent meetings, including an AMS/CMS meeting in Montréal in May 2002, a BIRS workshop in Banff in August 2004, and a CIRM workshop in Luminy in April 2005. It was suggested at the BIRS meeting that the draft text of M. Aguiar and S. Mahajan be expanded into the first monograph on the subject. Both are outstanding communicators. Their unified geometric approach using the combinatorics of Coxeter complexes and projection maps allows us to construct many of the combinatorial Hopf algebras currently under study and further to understand their properties (freeness, cofreeness, etc.) and to describe morphisms among them.

The current monograph is the result of this great effort and it is for me a great pleasure to introduce it.

Nantel Bergeron, Canada Research Chair, York University

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Preface

This research monograph deals with the interaction between the theory of Coxeter groups on one hand and the relationships among several Hopf algebras of recent interest on the other hand. It is aimed at upper-level graduate students and researchers in these areas. The viewpoint is new and leads to a lot of simplification.

0.1 The first part: Chapters 1-3

The first part, aside from Chapter 2, consists of standard material. The first two chapters are related to Coxeter theory, while the third chapter is related to Hopf algebras. We hope that they will make the second part more accessible.

Chapter 1 provides an introduction to some standard Coxeter theory written in a language suitable for our purposes. The emphasis is on the gate property and the projection maps of Tits, which are crucial in almost everything that we do. The reader may be required to accept many facts on faith, since most proofs are omitted. This chapter is a prerequisite for Chapter 5.

Chapter 2 is completely self-contained. It begins with some standard material on left regular bands (LRBs). We then develop some new material on pointed faces, lunes and bilinear forms on LRBs, largely inspired by the descent theory of Coxeter groups (Chapter 5). We also introduce the concept of a projection poset which generalizes the concept of an LRB to take into account some nonassociative examples.

Chapter 3 provides a brief discussion on cofree coalgebras, the coradical filtration and the antipode, which are standard notions in the theory of Hopf algebras. We then briefly discuss three examples of Hopf algebras which have now become standard: namely, the Hopf algebras of symmetric functions Λ, noncommutative symmetric functions $\mathrm{N}\Lambda$ and quasi-symmetric functions $\mathrm{Q}\Lambda$.

0.2 The second part: Chapters 4-8

The second part consists of mostly original work. The well-prepared reader may start directly with this part and refer back to the first part as necessary. Chapter 4 provides a brief overview of this work, which is spread over the next four chapters. Chapter 5 is related to Coxeter theory, while Chapters 6, 7 and 8 are related to Hopf algebras. Each of them is kept as self-contained as possible; the reader may even read them as different papers. A more detailed overview is given in the introductory section of each of these four chapters. The results in the second part, which are stated without credit, are new to our knowledge.

0.3 Future work

At many points in this monograph we say, "This will be explained in a future work." We plan to write a follow-up to this monograph, where these issues will be taken up. Our main motivation is not merely to prove new results or reprove existing results but rather to show that these ideas have a promising future.

0.4 Acknowledgements

We would like to acknowledge our debt to Jacques Tits, whose work provided the main foundation for this monograph. The work of Kenneth Brown on random walks and the literature on Hopf algebras, to which many mathematicians have contributed, provided us important guidelines. We would like to thank Nantel Bergeron for taking the initiative in having this work published, Carl Riehm and Thomas Salisbury for publishing this volume in the Fields monograph series, the referees for their comments and V. Nandagopal for providing TeX assistance.

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0.5 Notation

$\mathbb{K}$ stands for a field of characteristic 0. For P a set, we write $\mathbb{K}P$ for the vector space over $\mathbb{K}$ with basis the elements of P and $\mathbb{K}P^*$ for its dual space. A word is written in italics if it is being defined at that place. While looking for a particular concept, the reader is advised to search both the notation and the subject index. The notation $[n]$ stands for the set $\{1, 2, \ldots, n\}$. The table below indicates the main letter conventions that we use.

subsets	S, T, U, V
compositions	α, β, γ
partitions	λ, μ, ρ
faces or set compositions	F, G, H, K, N, P, Q
chambers	C, D, E
pointed faces or fully nested set compositions	(F, D), (P, C)
flats or set partitions	X, Y
lunes or nested set partitions	L, M

We write Σ for the set of faces and $\mathcal{C}$ for the set of chambers. Otherwise we use roman script for the above sets. For example, Q is the set of pointed faces, and L is the set of flats. For the coalgebras and algebras constructed from such sets, we use the calligraphic script $\mathcal{M}$, $\mathcal{N}$ and so on. There are some inevitable conflicts of notation; however, the context should keep things clear. For example, we also use the above letters F, M, K, H and S to denote various bases, V for a vector space, H for a Hopf algebra and S for an antipode.

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An important idea in the work of G.-C. Rota is that certain combinatorial objects give rise to Hopf algebras that reflect the manner in which these objects compose and decompose. Recent work has seen the emergence of several interesting Hopf algebras of this kind, which connect diverse subjects such as combinatorics, algebra, geometry, and theoretical physics. This monograph presents a novel geometric approach using Coxeter complexes and the projection maps of Tits for constructing and studying many of these objects as well as new ones. The first three chapters introduce the necessary background ideas making this work accessible to advanced graduate students. The later chapters culminate in a unified and conceptual construction of several Hopf algebras based on combinatorial objects which emerge naturally from the geometric viewpoint. This work lays a foundation and provides new insights for further development of the subject.

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