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Preface

The main purpose of the present notes is to give a systematic approach to conformal field theory with gauge symmetry, the so called the Wess-Zumino-Witten-Novikov model from the viewpoint of complex algebraic geometry. After presenting basic facts on the theory of compact Riemann surfaces, and on the representation theory of affine Lie algebras in Chapters 1 and 2, respectively, we shall construct conformal blocks for stable pointed curves with coordinates in Chapter 3. In Chapter 4 we shall construct the sheaf of conformal blocks associated to a family of stable pointed curves with coordinates. In Chapter 5 it will be shown that the sheaf of conformal blocks carries a projectively flat connection, which is one of the most important facts of conformal field theory. Chapter 6 is devoted to study the detailed structure of the conformal field theory over $\mathbb{P}^1$.

Recently J.E. Andersen and I constructed modular functors from conformal field theory. This gives an interesting relationship between Algebraic Geometry and Topological Quantum Field Theory. The present notes include all the necessary techniques and results on conformal field theory with gauge symmetry, which are used to construct the modular functor.

The present notes are based on the lectures and talks given at the Fields Institute, Queen's University, Århus University, Kobe University and Kyoto University. I thank the enthusiastic audiences who helped me to improve certain parts of the proofs in these notes.

My thanks are due to Noriko Yui for inviting me to write the present notes for Fields Institute Monographs and for helping me to complete the manuscript. Last but not least I would like to express my hearty thanks to Arthur Greenspoon of Mathematical Reviews for smoothing out the English of my original manuscript.

March 2008
Kenji Ueno

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Notation

S_ω the projective connection attached to a symmetric bidifferential ω (see (1.41)),

T_X the holomorphic (or regular) tangent bundle of a complex manifold (or smooth projective variety) X,

Θ_X the sheaf of holomorphic vector fields of a complex manifold or a smooth projective variety X,

Ω_X^k the sheaf of holomorphic k-forms of a complex manifold or a smooth projective variety X,

$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((\xi)) \oplus \mathbb{C} \cdot c$,

$\widehat{\mathfrak{g}}_N = \mathfrak{g} \otimes_{\mathbb{C}} \left(\bigoplus_{j=1}^N \mathbb{C}((\xi_j)) \right) \oplus \mathbb{C} \cdot c$,

Δ the root system of a complex simple Lie algebra and its Cartan subalgebra $(\mathfrak{g}, \mathfrak{h})$,

Δ_+ the set of positive roots,

θ the longest root of a complex simple Lie algebra $\mathfrak{g}$,

$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$,

$(\ ,\ )$ Cartan-Killing form of a complex simple Lie algebra $\mathfrak{g}$ normalized as $(\theta, \theta) = 2$,

P_+ the set of integral dominant weights of a complex simple Lie algebra $\mathfrak{g}$,

$P_\ell = \{\lambda \in P_+ \mid (\lambda, \theta) \leq \ell\}$,

$\mu^\dagger = -w(\mu)$ for $\mu \in P_\ell$ where w is the longest element of the Weyl group of a complex simple Lie algebra $\mathfrak{g}$,

$\kappa = g^* + \ell$ where g^* is the dual Coxeter number of a complex simple Lie algebra $\mathfrak{g}$, ($\kappa = n + 1 + \ell$ in case $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$),

$c_v = \frac{\ell \dim \mathfrak{g}}{\kappa}$,

$\Delta_\lambda = \frac{(\lambda, \lambda + 2\rho)}{2\kappa}$,

$\widehat{\Delta}_v = \Delta_\lambda + \Delta_{\mu_1} - \Delta_{\mu_2}$,

$\widehat{\Delta}_{\mathbf{v}} = \Delta_\lambda + \Delta_{\mu_1} - \Delta_{\mu_2}$ for a vertex $\mathbf{v} = \left({}_{\mu_1}{}^{\lambda}{}_{\mu_2} \right)$,

$X_n = \mathbb{C}^n \setminus \cup_{i<j} \Delta_{ij}, \quad \Delta_{ij} = \{(z_n, \ldots, z_1) \in \mathbb{C}^n \mid z_i = z_j\}$,

$\mathcal{R}_n = \{\, (z_n, \ldots, z_1) \in \mathbb{C}^n \mid |z_n| > |z_{n-1}| > \cdots > |z_1|\}$,

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Appendix

In this appendix we shall give basic results on the hypergeometric function. For any complex number α and for any non-negative integer n define

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)$$

where

$$(\alpha)_0 = 1.$$

Using the gamma function we can express $(\alpha)_n$ as

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$$

The hypergeometric series $F(\alpha,\beta,\gamma;z)$ is defined by

$$F(\alpha,\beta,\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} z^n = 1 + \frac{\alpha\beta}{\gamma} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)2!} z^2 + \cdots, \tag{A.39}$$

where α, β, γ are arbitrary complex numbers. The radius of convergence of the series $F(\alpha,\beta,\gamma;z)$ is 1, hence it defines a holomorphic function on the unit disk. The holomorphic function can be analytically extended to a multi-valued holomorphic function on $\mathbb{C}\setminus\{0,1\}$. We use the same notation $F(\alpha,\beta,\gamma;z)$ for the multi-valued holomorphic function and call it the hypergeometric function. The hypergeometric function $F(\alpha,\beta,\gamma;z)$ is a solution of the hypergeometric differential equation:

$$z(1-z)\frac{d^2u}{dz^2} + (\gamma - (\alpha+\beta+1)z)\frac{du}{dz} - \alpha\beta u = 0. \tag{A.40}$$

The hypergeometric differential equation (A.40) has regular singular points at 0, 1 and ∞. For simplicity in the following assume that γ is not a negative integer or 0. In a neighbourhood of 0 two linearly independent solutions are given by

$$\begin{aligned} u_{0,1} &= F(\alpha,\beta,\gamma;z) \\ u_{0,2} &= z^{1-\gamma}F(1-\gamma+\alpha, 1-\gamma+\beta, 2-\gamma; z). \end{aligned} \tag{A.41}$$

In a neighbourhood of 1 two linearly independent solutions are given by

$$u_{1,1} = F(\alpha,\beta,\alpha+\beta-\gamma+1; 1-z) \tag{A.42}$$

$$u_{1,2} = (1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta;z). \tag{A.43}$$

Similarly, in a neighbourhood of ∞ two linearly independent solutions are given by

$$u_{\infty,1} = (-z)^{-\alpha}F(\alpha, 1+\alpha-\gamma, 1+\alpha-\beta; \frac{1}{z}) \tag{A.44}$$

$$u_{\infty,2} = (-z)^{-\beta}F(\beta, 1+\beta-\gamma, 1+\beta-\alpha; \frac{1}{z}). \tag{A.45}$$

The analytic continuation of $F(\alpha, \beta, \gamma; z)$ in a neighbourhood of the point ∞ is given by

$$F(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)\Gamma(\beta - \alpha)}{\Gamma(\beta)\Gamma(\gamma - \alpha)} F(\alpha, 1 + \alpha - \gamma, 1 + \alpha - \beta; \frac{1}{z}) + \frac{\Gamma(\gamma)\Gamma(\alpha - \beta)}{\Gamma(\alpha)\Gamma(\gamma - \beta)} F(\beta, 1 + \beta - \gamma, 1 + \beta - \alpha; \frac{1}{z}) \qquad (A.46)$$

A proof can be found in Whittaker & Watson [WW] 14.5. This fact is used to calculate the connection matrix in Theorem 6.25.

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This book presents a systematic approach to conformal field theory with gauge symmetry from the point of view of complex algebraic geometry. After presenting the basic facts of the theory of compact Riemann surfaces and the representation theory of affine Lie algebras in Chapters 1 and 2, conformal blocks for pointed Riemann surfaces with coordinates are constructed in Chapter 3. In Chapter 4 the sheaf of conformal blocks associated to a family of pointed Riemann surfaces with coordinates is constructed, and in Chapter 5 it is shown that this sheaf supports a projective flat connection—one of the most important facts of conformal field theory. Chapter 6 is devoted to the study of the detailed structure of the conformal field theory over $\mathbb{P}^1$.

Recently it was shown that modular functors can be constructed from conformal field theory, giving an interesting relationship between algebraic geometry and topological quantum field theory. This book provides a timely introduction to an intensively studied topic of conformal field theory with gauge symmetry by a leading algebraic geometer, and includes all the necessary techniques and results that are used to construct the modular functor.

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