Polyhedral and Semidefinite Programming Methods in Combinatorial Optimization

Levent Tunçel
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The Fields Institute
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2010 Mathematics Subject Classification. Primary 90C22, 90C27, 15A39, 52A41, 65Y20, 05C69, 90C05, 90C25, 90C51, 68Q25.

For additional information and updates on this book, visit www.ams.org/bookpages/fim-27

Library of Congress Cataloging-in-Publication Data
Tunçel, Levent, 1965–
Polyhedral and semidefinite programming methods in combinatorial optimization / Levent Tunçel.
p. cm. — (Fields Institute monographs ; 27)
Includes bibliographical references and index.
ISBN 978-0-8218-3352-0 (alk. paper)
QA402.5.T86 2010
519.7—dc22
2010031316


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10 9 8 7 6 5 4 3 2 1 21 20 19 18 17 16
Contents

Preface ix

Chapter 1. Introduction 1
  1.1. Linear Programming 1
  1.2. Semidefinite Programming 9
  1.3. Fundamentals of Polyhedral Theory 19
  1.4. Further Bibliographical Notes 25

Chapter 2. Duality Theory 27
  2.1. Dual Cones 27
  2.2. Polars of (Compact) Sets 28
  2.3. Conjugates of (Convex) Functions 28
  2.4. A Strong Duality Theorem via the Hahn-Banach Theorem 29
  2.5. Linear Consequences, Proving Unboundedness, Strong Infeasibility Certificates 36
  2.6. Slater Condition, Borwein-Wolkowicz approach 39
  2.7. When does the Slater Condition Hold in SDP Relaxations? 43
  2.8. Bibliographical Notes 50

Chapter 3. Ellipsoid Method 53
  3.1. Ingredients: Separation Oracles, Inscribed and Circumscribed Ellipsoids 54
  3.2. Complexity Analysis 57
  3.3. Applications to SDP Problems 58
  3.4. Applications to Combinatorial Optimization Problems 59
  3.5. Equivalence of Separation and Optimization 60
  3.6. Bibliographical Notes 61

Chapter 4. Primal-Dual Interior-Point Methods 63
  4.1. Central Path 65
  4.2. Primal-Dual Potential Function 67
  4.3. Algorithm and Computational Complexity Analysis 68
  4.4. Auxiliary Self-Dual Problems 78
  4.5. Infeasible-Start Algorithms 79
  4.6. Other Interior-Point Algorithms, General Remarks 80
  4.7. Further Bibliographical Notes 80

Chapter 5. Approximation Algorithms Based on SDP 83
  5.1. MAX CUT, Goemans-Williamson Analysis 84
  5.2. Karloff’s Worst-Case Analysis 90
5.3. MAX 2SAT 91
5.4. Generalizations to Quadratic Optimization Problems 96
5.5. Further Bibliographical Notes 103

Chapter 6. Geometric Representations of Graphs 105
6.1. \( L_1 \) embeddability 105
6.2. Approximating the Sparsest Cuts via SDP 106
6.3. Unit Distance Representations of Graphs 109
6.4. Hypersphere Representations of Graphs 115
6.5. Orthonormal Representations of Graphs 116
6.6. Products of Graphs, Kronecker Products 122
6.7. Stable Set Problem and Shannon Capacity 123
6.8. Realizability of Graphs 126
6.9. Bibliographical Notes 126

Chapter 7. Lift-and-Project Procedures for Combinatorial Optimization Problems 129
7.1. Lovász-Schrijver Procedures 131
7.2. Balas-Ceria-Cornuéjols Procedure 135
7.3. Sherali-Adams (Reformulation-Linearization) Procedure 136
7.4. Optimization over Subset Lattices Interpretation 138
7.5. Bibliographical Notes 141

Chapter 8. Lift-and-Project Ranks for Combinatorial Optimization 143
8.1. Lower Bounds on the \( N_0 \)-Rank, \( N \)-Rank and \( N_+ \)-Rank 144
8.2. Matching Polytope and the Related Polyhedra 147
8.3. Stable Set Problem and Graph Ranks 152
8.4. Graph Ranks for Max Cut 160
8.5. TSP Polytope 161
8.6. Bibliographical Notes 161

Chapter 9. Successive Convex Relaxation Methods 165
9.1. Fundamental Framework 166
9.2. Discretized/Localized Method 172
9.3. Finite Convergence 173
9.4. Complexity Analysis 173
9.5. Applications to Systems of Polynomial Inequalities 175
9.6. Bibliographical Notes 175

Chapter 10. Connections to Other Areas of Mathematics 177

Chapter 11. An Application to Discrepancy Theory 183
11.1. Introduction to Discrepancy Theory via Optimization 183
11.2. Lovász-Sós’ Approach to Roth’s Theorem 184
11.3. Lovász’ SDP Approach to Roth’s Theorem 185
11.4. A Primal-Dual SDP Approach to Roth’s Theorem 188
11.5. Bibliographical Notes 188

Chapter 12. SDP Representability 189
12.1. Functions whose Epigraphs are SDP Representable 189
12.2. Generalized Epigraphs with respect to a Cone 199
## Contents

12.3. Representing Convex Cones as Feasible Regions of SDP Problems  200  
12.4. Bibliographical Notes  202  

Bibliography  203  

Index  217
Preface

Since the early 1960’s, polyhedral methods have had a central role to play in both the theory and practice of combinatorial optimization. Since the early 1990’s, a new technique, semidefinite programming, has been increasingly applied to some combinatorial optimization problems. The semidefinite programming problem is the problem of optimizing a linear function of matrix variables, subject to finitely many linear inequalities and the positive semidefiniteness condition on some of the matrix variables. On certain problems, such as maximum cut, maximum satisfiability, maximum stable set and geometric representations of graphs, semidefinite programming techniques yield important, new results. In this monograph, we provide the necessary background to work with semidefinite optimization techniques, usually by drawing parallels to the development of polyhedral techniques and with a special focus on combinatorial optimization, graph theory and lift-and-project methods.

The core of this monograph is based on ten lectures given at the Fields Institute during the academic term Fall-1999. This activity was a part of a special year of activities at the Fields Institute under the heading Graph Theory and Combinatorial Optimization. During the terms Fall-2001, Fall-2003, Spring-2005, Spring-2006, Spring-2007 as well as Fall-2008, I gave a course entitled Semidefinite Optimization at the Department of Combinatorics and Optimization, Faculty of Mathematics, University of Waterloo. During the course, I used and expanded some of the material from my Fields Institute lectures. These lecture notes and the handouts that I prepared evolved to the current monograph.

As prerequisites for this monograph, a solid background in mathematics at the undergraduate level and some exposure to linear optimization are required. Some familiarity with computational complexity theory and the analysis of algorithms would be helpful.

The chapters are exactly in the same order as the lectures. The first chapter familiarizes the audience with the basic concepts, notation, and lays down some theory to motivate the focus of the monograph, sometimes by way of analogy to the mainstream polyhedral approaches. Duality theory is paramount. As a result, instead of continuing with the material in Chapter 12 (which covers some examples of convex sets that can be represented as the feasible regions of Semidefinite Optimization problems) which would be the right way to go for an application-oriented audience, I took a risk and chose to cover duality theory as early as possible (Chapter 2). Then comes the theory of algorithms for convex optimization (Chapters 3 and 4). In Chapter 3, I give a quick overview of the Ellipsoid Method and in Chapter 4, I go through the theory of interior-point methods, with a focus on symmetric, primal-dual algorithms. This portion of the monograph (Chapters 1–4)
aims to establish rigorously most of the fundamental tools needed for Semidefinite Optimization. Chapters 5 and 6 cover various impressive results in Combinatorial Optimization and Graph Theory involving Semidefinite Optimization in a central way. Chapter 7 starts moving towards more abstract approaches in combinatorial optimization which use Semidefinite Optimization (Lift-and-Project Operators). Chapter 8 covers some of the basic techniques to analyze Lift-and-Project procedures with a special emphasis on the stable set problem. Chapter 9 considers yet further abstraction and generalization of these methods and prepares the audience for Chapter 10. The latter chapter is a collection of pointers to various wonderful results, some from other areas of mathematics and some establishing connections to such areas. Chapter 11 is a quick application to a cute theorem in number theory. Chapter 12 brings the lectures to a close in a nice, straightforward way with some obviously interesting open questions. Open problems of seemingly varying difficulty have been sprinkled throughout the text.

I thank Joseph Cheriyan for providing many interesting references over the years and Steven Karp, Graeme Kemkes, Lingchen Kong, Cristiane Sato, and Marcel Silva for many very useful remarks on earlier versions of the monograph. I also thank six anonymous referees for their very useful remarks and suggestions. I thank the editor Carl Riehm for his work and patience.

My research efforts were supported in part by Natural Sciences and Engineering Research Council of Canada, and by a Premier’s Research Excellence Award from Ontario, Canada. The support is gratefully acknowledged.

Levent Tunçel
Waterloo, Canada 2010
Bibliography


Bibliography


Bibliography


Index

K-moment problem, 179
K-moment sequences, 180
$L_2^2$-representation, 107
$L_p$-embeddability, 106
$M_2$-commutative, 149
$N$-rank, 143
$N^+\text{-rank}$, 143
$N^0\text{-rank}$, 143
$N\#\text{-commutative}$, 149
APX-hard, 83
S-Lemma, 37
$\delta$-relaxation, 54
c-balanced graph-separator problem, 108
adjoint of a linear transformation, 9
almost feasible, 36
antiblocker, 122
approximation algorithm, 83
$\mu$-approximation, 97
$p$-approximation, 83
arithmetic progressions, 184
arithmetic-geometric mean inequality, 66
arithmetic-harmonic mean inequality, 71
balanced matrix, 22
Balas-Ceria-Cornu`ejols procedure, 135
bisection method, 53
Boolean quadric polytope, 95
Carathéodory’s Theorem, 14
central path, 65
central surface, 79
Cholesky decomposition, 10
Chvátal rank, 25
circulant matrix, 185, 186
clique, 22
clique polytope (CLQ), 117
clique-node incidence matrix, 22
cloning a node, 159
closed half-space, 1
Colin de Verdière number of $G$, 127
compact representation, 129
cone
automorphism group of a cone, 14
dual cone, 13
homogeneous cone, 14
hyperbolic cone, 200
self-dual cone, 14
symmetric cone, 14
constraint qualification, 31
convex corner, 154
convex hull, 20
convex set, 4
pointed, 4
copositive programming relaxations, 162
covering problems, 22
cross-over algorithm, 80
cut, 84
cut cone, 106
cut norm of a matrix, 102
derandomization, 89
destruction of a node, 152
diagonally dominant, 13
dihedral group, 112
direct sum, 15
Dirichlet’s Lemma, 186
discrepancy of $x$, 183
discrepancy theory, 183
Discretized SSDPR algorithm, 172
disjunctive programming, 122
domain of a function, 66
eigenvalue, 9
eigenvector, 10
elementary arithmetic operations, 7
ellipsoid, 53
epigraph, 28
Euclidean distortion of $V$, 180
expander graphs, 108
extreme point, 4
extreme ray, 14
face, 39
exposed face, 39
projectionally exposed face, 39
proper face, 39
facet, 19
Farkas’ Lemma, 3
feasible solution, 1
First-Order Algorithms, 61
Fourier-Motzkin Elimination, 2
Index

Frobenius norm, 10

Gaussian Elimination, 1
generalized Lax conjecture, 200
Gershgorin Disk Theorem, 13
Goldman-Tucker Strict Complementarity Theorem, 34
Gomory-Chvátal cuts, 24
graph ranks, 152
Grothendieck’s constant, 103
Grothendieck’s inequality, 103
Hadamard product, 17
Hahn-Banach Separation Theorem, 29
Hamiltonian cycle, 59
Hankel matrix, 198
Heine-Borel characterization of compactness, 31
Hilbert’s 17th problem, 177
Hilbert’s Nullstellensatz, 179
Hirsch Conjecture, 7
homogeneous convex inequality form, 49
homogeneous equality form, 44
hub node, 153
Hyperbolic Feasibility Problem (HFP), 201
hyperbolic polynomial, 200
ideal matrix, 23
information complexity, 54
interior-point method, 65
iteration complexity, 8
Klee-Minty cube, 7
Kneser graph, 90
Kronecker product, 18
symmetric Kronecker product, 18
Löwner-John ellipsoid, 54
Laplacian matrix, 105
Legendre-Fenchel conjugate, 28, 66
lifted representation, 85
lifted space, 130
lifted-LMI representation, 201
lifted-SDP representation, 200
lifting, 129
line graph, 151
LMI representation, 201
Localized SSDPR algorithm, 172
Lovász-Schrijver Procedures, 131
LP, 1
Möbius matrix, 139
Markov chain, 108
conductance, 109
ergodic, 108
irreducible, 108
time-reversible, 109
matching polytope, 147
maximum weight matching problem, 147
matrix

leading principle minor of a matrix, 13
symmetric principle minor of a matrix, 11
matrix cube, 102
maximum cut problem (Max Cut), 84
maximum satisfiability (Max Sat), 91
MAXSNP, 83
MAXSNP-hard, 83
measure of centrality, 65
Minkowski sum, 27
moment matrix of f, 140
Motzkin’s example, 178
neighbourhood of a node, 152
network matrix, 21
node induced subgraph, 22
node-symmetric (vertex transitive), 111
node-symmetric(vertex-transitive), 126
nonapproximability threshold, 84
nonhomogeneous equality form, 48
objective function, 1
odd anti-hole, 23
odd subdivision of an edge, 156
odd-cycle polytope, 85
odd-hole, 23
odd-wheel inequality, 153
operator p-norm, 10
optimal solution, 1
optimum objective value, 1
orthonormal representation constraint, 118
orthonormal representations of graphs, 116
packing problems, 22
perfect graphs, 22
Strong Perfect Graph Theorem, 23
perfect matrices, 23
polar of a convex set, 28
polyhedron, 1
lower comprehensive, 154
pointed, 4
polynomial optimization problems (POP), 175
polynomial time algorithm, 7
positive semidefinite, 10
Positivstellensatz, 179
potential function, 67
potential-reduction algorithm, 74
primal-dual symmetry, 68
PSD-convex, 199
PSD-monotone, 199
purification algorithm, 80
quadratic effect, 78
rational polyhedron, 2
relative approximation ratio, 97
rim node, 153
sandwich theorem, 124
satisfiability problem (SAT), 91
satisfying assignment, 91
Index

scale-invariance, 68
Schur Complement Lemma, 17
SDP representation, 200
SDP-feasibility problem, 78
search direction, 68
AHO direction, 80
HKM direction, 80
NT direction, 80
Second Order Cone, 193
Second Order Cone Programming (SOCP) problem, 193
separating hypersphere theorem, 168
separation oracle, 54
separation problem for \( \Sigma^d \), 59
weak separation oracle, 54
separation theorem, 29
Shannon capacity, 123
Sherali-Adaams (RLT) Procedure, 136
Sherman-Morrison-Woodbury formula, 56
singular value decomposition, 88
Slater condition, 31
Slater point, 31
smallest hypersphere representation of graphs, 115
sparsest cut problem, 107
spectral gap, 109
stability number of a graph, 117
stable set, 22
stable set polytope STAB, 117
stable set problem, 117
stretching of a node, 157
strict complementarity, 34
strict complementarity for LP, 34
strict complementarity for SDP, 35
strictly convex function, 63
Strong Duality Theorem (SDP), 31
strong products of graphs, 122
strongly polynomial time algorithms, 8
subdivision of a graph, 156
subdivision of a star, 156
subgradient oracle, 58
submodular function minimization, 60
subset lattice interpretation, 138
subtour elimination polytope (SEP), 59
Successive SDP Relaxation (SSDPR), 168
sum of \( k \)-largest eigenvalues, 191
sum of squares (SoS), 177
sum off-largest singular values, 192
supermodularity, 140
symmetrized similarity transformation, 80
Taylor’s Theorem, 69
theta body for \( G \), 118
Total Dual Integrality (TDI), 24
Totally Unimodular (TUM), 21
trace, 9
traveling salesman problem (TSP), 59
tree-width of graphs, 162
triangle inequalities, 84
Turing Machine Model, 7
unit distance representation of graphs, 109
valid inequality, 19
volume of ellipsoid, 53
of unit ball, 53
zeta matrix, 138
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A solid background in mathematics at the undergraduate level and some exposure to linear optimization are required. Some familiarity with computational complexity theory and the analysis of algorithms would be helpful. Readers with these prerequisites will appreciate the important open problems and exciting new directions as well as new connections to other areas in mathematical sciences that the book provides.