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Polyhedral and Semidefinite Programming Methods in Combinatorial Optimization

Levent Tunçel

American Mathematical Society

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Preface

Since the early 1960's, polyhedral methods have had a central role to play in both the theory and practice of combinatorial optimization. Since the early 1990's, a new technique, semidefinite programming, has been increasingly applied to some combinatorial optimization problems. The semidefinite programming problem is the problem of optimizing a linear function of matrix variables, subject to finitely many linear inequalities and the positive semidefiniteness condition on some of the matrix variables. On certain problems, such as maximum cut, maximum satisfiability, maximum stable set and geometric representations of graphs, semidefinite programming techniques yield important, new results. In this monograph, we provide the necessary background to work with semidefinite optimization techniques, usually by drawing parallels to the development of polyhedral techniques and with a special focus on combinatorial optimization, graph theory and lift-and-project methods.

The core of this monograph is based on ten lectures given at the Fields Institute during the academic term Fall-1999. This activity was a part of a special year of activities at the Fields Institute under the heading *Graph Theory and Combinatorial Optimization.* During the terms Fall-2001, Fall-2003, Spring-2005, Spring-2006, Spring-2007 as well as Fall-2008, I gave a course entitled *Semidefinite Optimization* at the Department of Combinatorics and Optimization, Faculty of Mathematics, University of Waterloo. During the course, I used and expanded some of the material from my Fields Institute lectures. These lecture notes and the handouts that I prepared evolved to the current monograph.

As prerequisites for this monograph, a solid background in mathematics at the undergraduate level and some exposure to linear optimization are required. Some familiarity with computational complexity theory and the analysis of algorithms would be helpful.

The chapters are exactly in the same order as the lectures. The first chapter familiarizes the audience with the basic concepts, notation, and lays down some theory to motivate the focus of the monograph, sometimes by way of analogy to the mainstream polyhedral approaches. Duality theory is paramount. As a result, instead of continuing with the material in Chapter 12 (which covers some examples of convex sets that can be represented as the feasible regions of Semidefinite Optimization problems) which would be the right way to go for an application-oriented audience, I took a risk and chose to cover duality theory as early as possible (Chapter 2). Then comes the theory of algorithms for convex optimization (Chapters 3 and 4). In Chapter 3, I give a quick overview of the Ellipsoid Method and in Chapter 4, I go through the theory of interior-point methods, with a focus on symmetric, primal-dual algorithms. This portion of the monograph (Chapters 1–4) aims to establish rigorously most of the fundamental tools needed for Semidefinite Optimization. Chapters 5 and 6 cover various impressive results in Combinatorial Optimization and Graph Theory involving Semidefinite Optimization in a central way. Chapter 7 starts moving towards more abstract approaches in combinatorial optimization which use Semidefinite Optimization (Lift-and-Project Operators). Chapter 8 covers some of the basic techniques to analyze Lift-and-Project procedures with a special emphasis on the stable set problem. Chapter 9 considers yet further abstraction and generalization of these methods and prepares the audience for Chapter 10. The latter chapter is a collection of pointers to various wonderful results, some from other areas of mathematics and some establishing connections to such areas. Chapter 11 is a quick application to a cute theorem in number theory. Chapter 12 brings the lectures to a close in a nice, straightforward way with some obviously interesting open questions. Open problems of seemingly varying difficulty have been sprinkled throughout the text.

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Waterloo, Canada 2010

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