The General Topology of Dynamical Systems

Ethan Akin

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ABSTRACT. Recent work in smooth dynamical systems theory has highlighted certain topics from topological dynamics. This book organizes these ideas to provide the topological foundations for dynamical systems theory in general. The central theme is the importance of chain recurrence. The theory of attractors and different notions of recurrence and transitivity arise naturally as do various Lyapunov function constructions. The results are applied to the study of invariant measures and topological hyperbolicity.

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For Paul Akin
May 30, 1908 – May 26, 1992

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Preface

A large branch of modern dynamical systems theory has grown out of the work of Smale and his colleagues. The germinal technical concept, hyperbolicity, extended from a fixed point to more general invariant sets, consists of conditions imposed upon the tangent maps of the system. However, development of the subject revealed the fruitfulness of a number of purely topological concepts such as attractor, basic set, filtration, and chain recurrence. While some of these ideas were new, many were familiar objects of study in topological dynamics. The latter was a well established subject, flourishing, and somewhat separated from the differentiable theory. However, perusal of surveys like Bhatia and Szegö (1970) and Nemytskii and Stepanov (1960) reveals that topological dynamics drew much of its motivation, as well as many of its examples, from the still older qualitative theory of differential equations originating with Poincaré and exemplified in Andronov, Vitt, and Khaikin's great book (1937).

The recent global results associated with hyperbolicity have provided a new perspective on topological dynamics. For me this new view began with a look at Shub and Smale's 1972 paper, *Beyond hyperbolicity*. This book is the result of an often interrupted contemplation of the best way to organize the parts of topological dynamics which are most useful for the nonspecialist. John Kelley wrote in the preface to his justly famous book, *General topology*, that he was, with difficulty, prevented by his friends from using the title "What every young analyst should know". The reader will note that I have adapted his title. This is partly gratitude (and an attempt at sympathetic magic), but mostly because my intent is inspired by his. I hope to have described what every dynamicist should know, or at least be acquainted with, from topological dynamics.

While the book is thus intended as a service text and reference, its subject eventually organized itself into a unified story whose central theme is the role of chain recurrence in the study of dynamical systems on compact metric spaces. The assumption of metrizability is, for most of the results, just a convenience, but compactness is essential. We repeatedly use the preservation of compactness by continuous maps. Even more often we need the observation that for a decreasing sequence of nonempty compact sets, $\{A_n\}$, the intersection, A, is nonempty and if U is any neighborhood of A, then $A_n \subset U$ for sufficiently large n. On the other hand, we study the iterations not just of continuous maps but of more general closed relations on the space. At first glance, this appears to be one of those tedious and mechanical generalizations more honored in the omission than in the transcribing. Instead, even the homeomorphisms which are our primary interest are best studied by thickening them up to relations in various ways (Joseph Auslander's prolongations). Also, the relation results can be used to partly mitigate the unfortunate demand for compactness. Given a homeomorphism of a locally compact space we can restrict to a large compact subset, A. Of course, if A is an invariant set then the restriction is still a homeomorphism. But even if A is not even positive invariant, the restriction is a closed relation on A, though not a mapping. Furthermore, the relation results yield constructions on A, e.g., Lyapunov functions, more powerful than would be obtained by a further restriction to the largest invariant subset of A.

As for prerequisites, except for the measure theory in Chapter 8 and occasional forays into differentiable territory, what is needed is fluency in the topology of metric spaces. However, a reader whose background includes a modern treatment of differential equations like Hirsch and Smale (1974) or Arnold (1973) will have a better understanding of why we take up the topics that we do.

In Chapters 1–3 we develop the fundamentals of the dynamics of a closed relation. We introduce and apply various kinds of recurrence and invariant sets, the theory of attractors, and the construction of Lyapunov functions. With Chapters 4 and 5 we return to mappings to discuss topological transitivity, minimal subsets, decompositions and constructions converging upon the chain recurrent set. In Chapter 6 we derive the related results for flows and obtain special results for Lyapunov functions and chain recurrence in the vector field case. Chapter 7 concerns perturbation theory. Since our perturbations are topological rather than differentiable, the structural stability results associated with hyperbolicity do not apply, but we describe Takens' results on Zeeman's "tolerance stability conjecture". In Chapter 8 we describe invariant measures and compare topological notions of ergodicity and mixing with the measure theoretic versions. In Chapter 9 we apply the results to some important examples, e.g., shift maps on spaces of symbols and flows on the torus. Finally, in Chapters 10 and 11 we describe the hyperbolicity results for fixed points and for Axiom A homeomorphisms, respectively. This latter is the topological generalization of Smale's differential idea.

The results from the exercises in the text are used as lemmas and so should at least be read. The straightforward proofs are better performed by the reader (guided by the hints) or omitted entirely than laid out in detail on the printed page.

The second printing has provided an opportunity to correct some mistakes in the text and omissions in the references.

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Subject Index

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It contains a wealth of information concerning topological dynamics, most of which has not appeared before in such an organization and presentation. It offers to a graduate-level student a very comprehensive overview on the basic concepts in the theory of dynamical systems.

—Zentralblatt MATH

No other single text has heretofore presented such a unified treatment of these topological ideas at this level of generality.

—Mathematical Reviews

Topology, the foundation of modern analysis, arose historically as a way to organize ideas like compactness and connectedness which had emerged from analysis. Similarly, recent work in dynamical systems theory has both highlighted certain topics in the pre-existing subject of topological dynamics (such as the construction of Lyapunov functions and various notions of stability) and also generated new concepts and results (such as attractors, chain recurrence, and basic sets). This book collects these results, both old and new, and organizes them into a natural foundation for all aspects of dynamical systems theory. No existing book is comparable in content or scope. Requiring background in point-set topology and some degree of "mathematical sophistication", Akin's book serves as an excellent textbook for a graduate course in dynamical systems theory. In addition, Akin's reorganization of previously scattered results makes this book of interest to mathematicians and other researchers who use dynamical systems in their work.

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