

# The General Topology of Dynamical Systems

**Ethan Akin**

**Graduate Studies  
in Mathematics**

**Volume I**



**American Mathematical Society**

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## Editorial Board

Ronald R. Coifman  
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**ABSTRACT.** Recent work in smooth dynamical systems theory has highlighted certain topics from topological dynamics. This book organizes these ideas to provide the topological foundations for dynamical systems theory in general. The central theme is the importance of chain recurrence. The theory of attractors and different notions of recurrence and transitivity arise naturally as do various Lyapunov function constructions. The results are applied to the study of invariant measures and topological hyperbolicity.

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*For Paul Akin*  
*May 30, 1908 – May 26, 1992*

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## Preface

A large branch of modern dynamical systems theory has grown out of the work of Smale and his colleagues. The germinal technical concept, hyperbolicity, extended from a fixed point to more general invariant sets, consists of conditions imposed upon the tangent maps of the system. However, development of the subject revealed the fruitfulness of a number of purely topological concepts such as attractor, basic set, filtration, and chain recurrence. While some of these ideas were new, many were familiar objects of study in topological dynamics. The latter was a well established subject, flourishing, and somewhat separated from the differentiable theory. However, perusal of surveys like Bhatia and Szegö (1970) and Nemytskii and Stepanov (1960) reveals that topological dynamics drew much of its motivation, as well as many of its examples, from the still older qualitative theory of differential equations originating with Poincaré and exemplified in Andronov, Vitt, and Khaikin's great book (1937).

The recent global results associated with hyperbolicity have provided a new perspective on topological dynamics. For me this new view began with a look at Shub and Smale's 1972 paper, *Beyond hyperbolicity*. This book is the result of an often interrupted contemplation of the best way to organize the parts of topological dynamics which are most useful for the nonspecialist. John Kelley wrote in the preface to his justly famous book, *General topology*, that he was, with difficulty, prevented by his friends from using the title "What every young analyst should know". The reader will note that I have adapted his title. This is partly gratitude (and an attempt at sympathetic magic), but mostly because my intent is inspired by his. I hope to have described what every dynamicist should know, or at least be acquainted with, from topological dynamics.

While the book is thus intended as a service text and reference, its subject eventually organized itself into a unified story whose central theme is the role of chain recurrence in the study of dynamical systems on compact metric spaces. The assumption of metrizability is, for most of the results, just a convenience, but compactness is essential. We repeatedly use the

preservation of compactness by continuous maps. Even more often we need the observation that for a decreasing sequence of nonempty compact sets,  $\{A_n\}$ , the intersection,  $A$ , is nonempty and if  $U$  is any neighborhood of  $A$ , then  $A_n \subset U$  for sufficiently large  $n$ . On the other hand, we study the iterations not just of continuous maps but of more general closed relations on the space. At first glance, this appears to be one of those tedious and mechanical generalizations more honored in the omission than in the transcribing. Instead, even the homeomorphisms which are our primary interest are best studied by thickening them up to relations in various ways (Joseph Auslander's prolongations). Also, the relation results can be used to partly mitigate the unfortunate demand for compactness. Given a homeomorphism of a locally compact space we can restrict to a large compact subset,  $A$ . Of course, if  $A$  is an invariant set then the restriction is still a homeomorphism. But even if  $A$  is not even positive invariant, the restriction is a closed relation on  $A$ , though not a mapping. Furthermore, the relation results yield constructions on  $A$ , e.g., Lyapunov functions, more powerful than would be obtained by a further restriction to the largest invariant subset of  $A$ .

As for prerequisites, except for the measure theory in Chapter 8 and occasional forays into differentiable territory, what is needed is fluency in the topology of metric spaces. However, a reader whose background includes a modern treatment of differential equations like Hirsch and Smale (1974) or Arnold (1973) will have a better understanding of why we take up the topics that we do.

In Chapters 1–3 we develop the fundamentals of the dynamics of a closed relation. We introduce and apply various kinds of recurrence and invariant sets, the theory of attractors, and the construction of Lyapunov functions. With Chapters 4 and 5 we return to mappings to discuss topological transitivity, minimal subsets, decompositions and constructions converging upon the chain recurrent set. In Chapter 6 we derive the related results for flows and obtain special results for Lyapunov functions and chain recurrence in the vector field case. Chapter 7 concerns perturbation theory. Since our perturbations are topological rather than differentiable, the structural stability results associated with hyperbolicity do not apply, but we describe Takens' results on Zeeman's "tolerance stability conjecture". In Chapter 8 we describe invariant measures and compare topological notions of ergodicity and mixing with the measure theoretic versions. In Chapter 9 we apply the results to some important examples, e.g., shift maps on spaces of symbols and flows on the torus. Finally, in Chapters 10 and 11 we describe the hyperbolicity results for fixed points and for Axiom A homeomorphisms, respectively. This latter is the topological generalization of Smale's differential idea.

The results from the exercises in the text are used as lemmas and so should at least be read. The straightforward proofs are better performed by the reader (guided by the hints) or omitted entirely than laid out in detail on the printed page.

The second printing has provided an opportunity to correct some mistakes in the text and omissions in the references.

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## References

- E. Akin, *The metric theory of Banach manifolds*, Lecture Notes in Math., vol. 662, Springer-Verlag, Berlin and New York, 1978.
- D. Anosov, I. Bronstein, S. Aronson and V. Grines, *Smooth dynamical systems*, Dynamical Systems I (D. Anosov and V. Arnold, eds.), Encyclopedia of Mathematical Sciences, vol. 1, Springer-Verlag, Berlin and New York, 1988, 148–233.
- A. A. Andronov, A. A. Vitt, and S. E. Khaikin, *Theory of oscillations* (reprint of 1966 translation of 1937 second Russian edition), Dover, New York, 1987.
- V. I. Arnold, *Ordinary differential equations*, MIT Press, Cambridge, MA, 1973.
- J.-F. Aubin and H. Frankowska, *Set valued analysis*, Birkhäuser, Boston, 1990.
- J. Auslander, *Generalized recurrence in dynamical systems*, Contributions to Differential Equations, vol. 3, Wiley, New York, 1964, 55–74.
- , *Minimal flows and their extensions*, North-Holland, Amsterdam, 1988.
- J. Auslander, N. Bhatia and P. Siebert, *Attractors in dynamical systems*, Bol. Soc. Mat. Mex. **9** (1964), 55–66.
- J. Auslander and P. Siebert, *Prolongations and stability in dynamical systems*, Ann. Inst. Fourier, Grenoble **14** (1964), 237–268.
- M. Barnsley, *Fractals everywhere*, Academic Press, San Diego, CA, 1988.
- A. Beck, *On invariant sets*, Ann. of Math. **67** (1958), 99–103.
- N. Bhatia, *Weak attractors in dynamical systems*, Bol. Soc. Mat. Mex. **11** (1966), 56–64.
- , *On asymptotic stability in dynamical systems*, Math. Systems Theory **1** (1967), 113–127.
- , *Semidynamical flow near a compact invariant set*, Topological Dynamics (J. Auslander and W. Gottschalk, eds.), W. A. Benjamin, Inc., New York, 1968, 81–95.
- N. Bhatia, A. Lazer and G. Szegő, *On global weak attractors in dynamical systems*, J. Math. Anal. App. **16** (1966), 544–552.
- N. Bhatia and G. Szegő, *Stability theory of dynamical systems*, Springer-Verlag, Berlin and New York, 1970.
- , *Dynamical systems: stability theory and applications*, Lecture Notes in Math., vol. 35, Springer-Verlag, Berlin and New York, 1967.
- P. Billingsley, *Ergodic theory and information*, Wiley, New York, 1965.
- G. Birkhoff, *Dynamical systems*, Amer. Math. Soc., Providence, 1927.
- G. Birkhoff and P. Smith, *Structure analysis of surface transformations*, J. Math. Pure Appl. **7** (1928), 345–379.
- R. P. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture Notes in Math., vol. 470, Springer-Verlag, Berlin and New York, 1975.
- J. Brown, *Ergodic theory and topological dynamics*, Academic Press, New York,

- 1976.
- G. Butler, H. Freedman and P. Waldman, *Uniformly persistent systems*, Proc. Amer. Math. Soc. **96** (1986), 425–430.
- G. Butler and P. Waldman, *Persistence in dynamical systems*, J. Diff. Eq. **63** (1986), 255–263.
- C. Conley, *Some abstract properties of the set of invariant sets of a flow*, Illinois J. Math. **16** (1972), 663–668.
- , *Isolated invariant sets and the Morse index*, CBMS Regional Conf. Ser. in Math., vol. 38, Amer. Math. Soc., Providence, RI, 1978.
- E. Coven, J. Madden and Z. Nitecki, *A note on generic properties of continuous maps*, Ergodic Theory and Dynamical Systems II, Progress in Math. vol. 21, Birkhäuser, Boston, 1982, 97–101.
- M. Denker, C. Grillenberger and K. Sigmund, *Ergodic theory on compact spaces*, Lecture Notes in Math., vol. 527, Springer-Verlag, Berlin and New York, 1970.
- V. Dobrynsky and A. Sharkovsky, *Genericity of dynamical systems for which almost all trajectories are stable with respect to permanent perturbation*, Dokl. Acad. Nauk. SSSR **211** (1973), 273–276.
- Y. Dowker, *The mean and transitive points of homeomorphisms*, Ann. of Math. **58** (1953), 123–133.
- N. Fenichel, *Hyperbolicity conditions for dynamical systems*, preprint, 1975.
- J. E. Franke and J. F. Selgrade, *Hyperbolicity and chain recurrence*, J. Differential Equations **26** (1977), 27–36.
- H. Furstenberg, *Disjointness in ergodic theory, minimal sets and a problem in Diophantine approximation*, Math. Systems Theory **1** (1967), 1–49.
- , *Recurrence in ergodic theory and combinatorial number theory*, Princeton Univ. Press, Princeton, NJ, 1981.
- B. Garay, *Uniform persistence and chain recurrence*, J. Math. Anal. Appl. **139** (1989), 372–381.
- A. Gaunersdorfer, *Time averages for heteroclinic attractors*, SIAM J. Appl. Math. **52** (1992), 1476–1489.
- P. Halmos, *Measure theory*, Van Nostrand, Princeton, NJ, 1950.
- P. Hartman, *Ordinary differential equations*, Wiley, New York, 1964.
- M. Hirsch and C. Pugh, *Stable manifolds and hyperbolic sets*, Global Analysis (S-S. Chern and S. Smale, eds.), Proc. Sympos. Pure Math. vol. XIV, Amer. Math. Soc., Providence, RI, 1970, 133–163.
- M. Hirsch and S. Smale, *Differential equations dynamical systems, and linear algebra*, Academic Press, San Diego, CA, 1974.
- J. Hofbauer, *A unified approach to persistence*, Acta Applicanda Math. **14** (1989), 11–22.
- J. Hofbauer and K. Sigmund, *The theory of evolution and dynamical systems*, Cambridge Univ. Press, Cambridge, 1988.
- J. Hofbauer and J. So, *Uniform persistence and repellers for maps*, Proc. Amer. Math. Soc. **107** (1989), 1137–1142.
- R. B. Holmes, *A formula for the spectral radius of an operator*, MAA Monthly **75** (1968), 163–166.
- C. Hsu, *Cell-to-cell mapping*, Springer-Verlag, Berlin and New York, 1987.
- P. Huber, *Robust statistics*, Wiley, New York, 1981.
- M. Hurley, *Attractors: persistence and density of the basins*, Trans. Amer. Math. Soc. **269** (1982), 247–271.

- , *Bifurcation and chain recurrence*, Ergod. Th. and Dyn. Sys. **3** (1983), 231–240.
- , *Consequences of topological stability*, J. Diff. Eq. **54** (1984), 60–72.
- J. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), 713–747.
- M. C. Irwin, *Smooth dynamical systems*, Academic Press, San Diego, CA, 1980.
- J. Kelley, *General topology*, Van Nostrand, Princeton, NJ, 1955.
- J. C. Lagarias, *The  $3X + 1$  Problem and its generalizations*, MAA Monthly **92** (1985), 3–21.
- S. Lang, *Differential manifolds*, Addison-Wesley, Reading, MA, 1972.
- R. McGehee, *Attractors for closed relations on compact Hausdorff spaces*, Indiana Univ. Math. J. **41** (1992), 1165–1209.
- A. Maier, *On central trajectories and Birkhoff's problem*, Mat. Sbornik N.S. **26** (1950), 266–290.
- R. Mañé, *Ergodic theory and differentiable dynamics*, Springer-Verlag, Berlin and New York, 1983.
- E. Nelson, *Topics in dynamics: I. flows*, Princeton Univ. Press, Princeton, NJ, 1969.
- V. V. Nemytskii and V. V. Stepanov, *Qualitative theory of differential equations*, Princeton Univ. Press, Princeton, NJ, 1960.
- S. Newhouse, *Lectures on dynamical systems*, Dynamical Systems. C.I.M.E. Lectures, Bressanone, Italy, June, 1978, Birkhäuser, Boston, 1980, 1–115.
- Z. Nitecki, *Differentiable dynamics*, MIT Press, Cambridge, MA, 1971.
- Z. Nitecki and M. Shub, *Filtrations, decompositions and explosions*, Amer. J. Math. **107** (1975), 1029–1048.
- J. Oxtoby, *Ergodic sets*, Bull. Amer. Math. Soc. **58** (1952), 116–136.
- , *Measure and category* (2nd Ed.), Springer-Verlag, Berlin and New York, 1980.
- J. Palis, C. Pugh, M. Shub, and D. Sullivan, *Genericity theorems in topological dynamics*, Dynamical Systems-Warwick, 1974, Lecture Notes in Math., vol. 468, Springer-Verlag, Berlin and New York, 1975, 234–240.
- S. Pilyugin, *The space of dynamical systems with the  $C^0$ -topology*, Lecture Notes in Math, vol. 1571, Springer-Verlag, Berlin and New York, 1994.
- C. Reed, *Two methods of defining the center of a dynamical system*, Topological Dynamics (J. Auslander and W. Gottschalk, eds.), W. A. Benjamin, Inc., New York, 1968, 391–399.
- G. Sell, *Topological dynamics and ordinary differential equations*, Van Nostrand Rheinhold Co., London, 1971.
- M. Shub and S. Smale, *Beyond hyperbolicity*, Ann. of Math. **96** (1972), 587–591.
- M. Shub, *Global stability of dynamical systems*, Springer-Verlag, Berlin and New York, 1987.
- P. Siebert, *Attractors in general systems*, Differential Equations-Oklahoma State University, 1979 (S. Ahmad, M. Keener and A. Lazer, eds.), Academic Press, New York, 1980, 249–270.
- S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817.
- , *The  $\Omega$ -stability theorem*, Global Analysis, (S-S. Chern and S. Smale, eds.), Proc. Sympos. Pure Math, vol. XIV, Amer. Math. Soc., Providence, RI, 1970, 289–297.

- , *The mathematics of time*, Springer-Verlag, Berlin and New York, 1980.
- G. Szegő, *Topological properties of weak attractors*, Topological Dynamics (J. Auslander and W. Gottschalk, eds.), W. A. Benjamin, Inc., New York, 1968, 455–467.
- F. Takens, *On Zeeman's tolerance stability conjecture*, Manifolds-Amsterdam 1970, Lecture Notes in Math., vol. 197, Springer-Verlag, Berlin and New York, 1971, 209–219.
- , *Tolerance stability*, Dynamical Systems-Warwick 1974, Lecture Notes in Math., vol. 468, Springer-Verlag, Berlin and New York, 1975, 293–304.
- T. Ura and I. Kimura, *Sur le courant extérieur à une région invariante, théorème de Bendixson*, Comm. Math. Univ. Sancti Pauli 8 (1960), 23–29.
- P. Walters, *On the pseudo-orbit tracing property and its relation to stability*, The Structure of Attractors in Dynamical Systems (N. Markley, J. Martin and W. Perrizo, eds.), Lecture Notes in Math., vol. 668, Springer-Verlag, Berlin and New York, 1978, 231–244.
- W. White, *On the tolerance stability conjecture*, Dynamical Systems (M. Peixoto ed.), Academic Press, New York, 1973, 663–665.
- K. Yosida, *Functional analysis*, Springer-Verlag, Berlin and New York, 1965.



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*It contains a wealth of information concerning topological dynamics, most of which has not appeared before in such an organization and presentation. It offers to a graduate-level student a very comprehensive overview on the basic concepts in the theory of dynamical systems.*

—**Zentralblatt MATH**

*No other single text has heretofore presented such a unified treatment of these topological ideas at this level of generality.*

—**Mathematical Reviews**

Topology, the foundation of modern analysis, arose historically as a way to organize ideas like compactness and connectedness which had emerged from analysis. Similarly, recent work in dynamical systems theory has both highlighted certain topics in the pre-existing subject of topological dynamics (such as the construction of Lyapunov functions and various notions of stability) and also generated new concepts and results (such as attractors, chain recurrence, and basic sets). This book collects these results, both old and new, and organizes them into a natural foundation for all aspects of dynamical systems theory. No existing book is comparable in content or scope. Requiring background in point-set topology and some degree of “mathematical sophistication”, Akin’s book serves as an excellent textbook for a graduate course in dynamical systems theory. In addition, Akin’s reorganization of previously scattered results makes this book of interest to mathematicians and other researchers who use dynamical systems in their work.

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