Graduate Studies in Mathematics

Volume 2

Combinatorial Rigidity

Jack Graver
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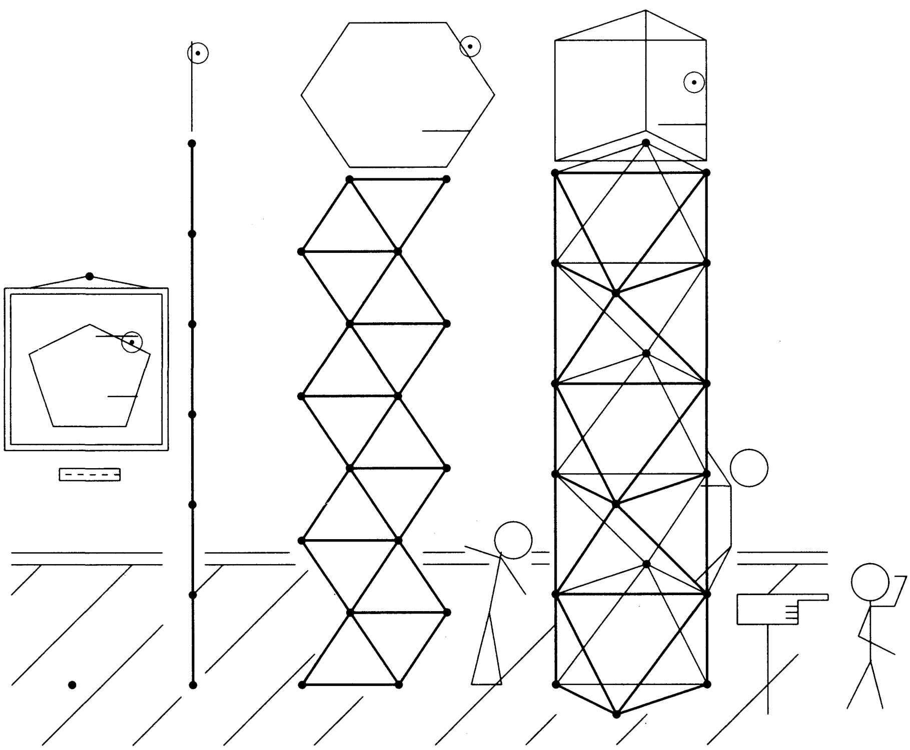

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ABSTRACT. Rigidity theory is introduced in an historical context. The combinatorial aspects of rigidity are isolated and framed in terms of a special class of matriods. These matriods are a natural generalization of the connectivity matriod of a graph. This book includes an introduction to matriod theory and a comrehensive study of planar rigidity. The final chapter of the text is devoted to higher dimensional rigidity, highlighting the main questions still open. This book contains an extensive annotated bibliography.

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Contents

Preface

A *framework* in m-space is a triple $(V, E, \mathbf{p})$, where (V, E) is a finite graph and $\mathbf{p}$ is an embedding of V into real m-space. A framework is a mathematical model for a physical structure in which each vertex v corresponds to an idealized ball joint located at $\mathbf{p}(v)$, and each edge corresponds to a rigid rod connecting the joints corresponding to its endpoints. Obviously this concrete realization is meaningful only for $m \leq 3$, and may be used to describe a very general class of physical structures, including rigid ones such as pedestals or bridges, as well as moving structures such machines or organic molecules. Making the distinction between frameworks whose realization is rigid and those which can move is the fundamental problem of rigidity theory, which can also be considered for frameworks in higher dimensions. In low dimensions one could construct an appropriate realization of a given framework and test the model for rigidity. Of course, the mathematical task is to develop a method for predicting rigidity without building a model.

One would expect that whether a framework is rigid or not depends on both the graph (V, E) and the embedding $\mathbf{p}$; or, in more general terms, that the question of rigidity has both combinatorial and geometric aspects. Our primary interest is in the combinatorial part of rigidity theory, which we call *combinatorial rigidity*. However, the two parts of rigidity theory are not so easily separated. In fact, only in dimensions one and two has total separation of the two parts been achieved.

In the first chapter we will give an overview of the subject, developing both aspects of the theory of rigidity informally in an historical context. This chapter stands apart from the rest of the book in that it contains no formal proofs. Most of the concepts introduced here will be reintroduced in a more formal setting later on.

The second chapter is devoted to a study of infinitesimal rigidity, a linear approximation which stands at the boundary of the combinatorial and geometric nature of rigidity. The infinitesimal approach offers at least a partial separation of the combinatorial and geometric aspects by regarding the matrix of the derivative of a framework motion as a matroid on the edges of the framework.

In general, depending on the dimension and the embedding, the edges of a graph are the underlying set of several such matroids, all of which belong to the class of abstract rigidity matroids, which are defined at end of chapter 2.

The fundamental combinatorial structures used to study rigidity are the various rigidity matroids. The second chapter consists of a development of matroid theory, the theoretical foundation for much of modern combinatorics. There will of course be a special emphasis on those parts of the subject most applicable to rigidity matroids.

Chapter 4 is devoted to an extensive study of combinatorial rigidity dimension 2, which has a nice analogy, via the 1-dimensional case, with "traditional" graph theory from a slightly different point of view. A thorough knowledge of planar rigidity is essential to developing a good intuition for rigidity as a whole, and provides an extensive collection of tractable examples. Algorithmic and computational aspects are also treated.

In the last chapter, we will discuss combinatorial rigidity in higher dimensions. Special attention is paid to dimension 3, in which there is the most practical interest, but where the characterization problem is still unsolved. Many of the results in this chapter have not yet appeared elsewhere.

The book concludes with an extensive annotated bibliography.

This text is suitable for a second graduate course in combinatorics and was already used as such at Syracuse University and at Worcester Polytechnic Institute by the authors. Each chapter contains a variety of exercises, some letting the reader fill in the details of the theory, some working through examples, as well as many which point the way to aspects of rigidity theory not covered in the text. Exercises are placed so that the reader can check his understanding of each concept before going on to the next one. The annotations in the bibliography are not only a valuable research tool, but also meant to stimulate a project oriented course of study.

Each of the chapters is mathematically self-contained, and the reader may safely peruse them in the order best suited to his background and interest.

Many thanks to Ray Adams and John Shutt for diligent proof reading and thoughtful questions.

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References

1. A.D. Alexandrov, (1950). *Convex polyhedra* (in Russian), Moskow.
German translation: *Convexe Polyeder.* In Chapter 10 the author proves that if vertices are added to the edges of a strictly convex polygon and the faces are then triangulated, the resulting 1-skeleton is infinitesimally rigid, and hence rigid. (This description is taken from [**2**].)
2. L. Asimow and B. Roth, (1978). *Rigidity of graphs,* Trans. Amer. Math. Soc. **245**, 279–289.
Introduces rigidity from the point of view of algebraic geometry. Proves that if $|E| < m|V| - (m+1)m/2$, then (V, E) is not generically rigid, as well as the fact that complete graphs are the only graphs rigid in all embeddings in all dimensions.
3. L. Asimow and B. Roth, (1979). *Rigidity of graphs II,* SIAM J. Appl. Math. **68** 1, 171–190.
Introduces infinitesimal rigidity. Defines *structures* (rods, panels, etc.), and *general position* (generic for all edge sets). Proves that infinitesimal rigidity, generic rigidity, and rigidity are equivalent if the vertices are in general position, in this sense. Proves for dimension 2: 0 and 1-extensions of independent sets are independent. If E_1 and E_2 are rigid edge sets and have 2 common vertices, then $E_1 \cup E_2$ is rigid. If E_1, E_2 and E_3 are rigid and if each pair has a distinct point in common, then $E_1 \cup E_2 \cup E_3$ is rigid. A totally dependent set (cycle) D contains $2|V(D)|-2$ elements any $|D|-1$ of which are independent and rigid. Proves for dimension 3: Alexandrov's extension of Cauchy's Theorem, Gluck's Theorem, and that independent sets satisfy Laman's Condition.
4. J. A. Baglivo and J. E. Graver, (1983). *Incidence and symmetry in design and architecture,* Cambridge Univ. Press.
Contains an exposition of the grid bracing problem.
5. D. Barnet and B. Grünbaum, (1969). *On Steinitz's Theorem concerning convex 3-polytopes and on some properties of planar graphs,* in "Many

Facets of Graph Theory", Springer Lecture Notes **110**, 27–40.

This paper is devoted to two proofs of Steinitz's Theorem: A graph is isomorphic to the 1-skeleton of some convex polyhedron in 3-space if and only if the graph is planar and 3-connected.

6. G. T. Bennett, (1912). *Deformable octahedra,* Proc. London Math. Soc. (2) **10**, 309–343.
7. W. Blaschke, (1920). *Wackelige Achtflache,* Math. Z. **6**, 85–93.
8. E. Bolker, (1979). *Bracing rectangular frameworks II,* SIAM J. Appl. Math. **36**, 491–508.

   This paper describes the minimal redundant sets of diagonal braces in a grid of cubes in space. No proofs.
9. E. Bolker, (1977). *Bracing grids of cubes,* Environment and Planning B **4**, 157–172.

   Simplified version of [**8**].
10. E. Bolker and H. Crapo, (1977). *How to brace a one story building,* Environment and Planning B **4**, 125–152.

    Simplified version of [**11**].
11. E. Bolker and H. Crapo, (1979). *Bracing rectangular frameworks,* Siam J. Appl. Math. **36**, 473–490.

    An introduction to infinitesimal rigidity applied to the grid bracing problem. Shows equivalence to bipartite graph model. Solves bracing problem for a planar grid in space supported by vertical rods of uniform length from a solid floor. (Contains [**28**].)
12. E. Bolker and B. Roth, (1980). *When is a bipartite graph a rigid framework?,* Pacific J. Math. **90**, 27–44.

    The relationship between infinitesimal rigidity and the stress space of a framework is studied in the special case that the framework is bipartite. The following results are obtained: $K_{4,n}$ (for $n > 5$) and $K_{m,n}$ (for $m, n > 4$) are generically rigid in 3-space. $K_{6,7}$ has enough edges to be generically rigid in 4-space but is not. A $K_{3,3}$ framework in the plane is infinitesimally rigid unless the vertices lie on a conic. A $K_{d+1,d(d+1)/2}$ framework in d-space such that each vertex set spans d-space is infinitesimally rigid unless the vertices lie on a conic. Several examples are discussed. (See also [**135**] & [**139**].)
13. O. Bottema, (1967). *Flexible hexagons,* Indigat. Math. **29**, 151–155.

    Shows that there are classes of non-singular flexible hexagons the opposite elements of which are not equal. (See also [**66**].)
14. R. Bricard, (1895). *Réponse à Question 376,* Intermédiaire Recherches Math. **2**, 245.

15. R. Bricard, (1897). *Mémoir sur la théorie de l'octaèdre articulé,* J. Math. Pure and Appl. **5** 3, 113–148.

A complete study of the embeddings of the 1-skeleton of the octahedron in 3-space. Two main results: All embeddings which extend to embeddings of the 2-skeleton are rigid. There is an embedding of the 1-skeleton which is not rigid.

16. C. R. Calladine, (1978). *Buckminster Fuller's "tensegrity" structures and Clerk Maxwell's rules for the construction of stiff frameworks,* Internat. J. Solids and Structures **14**, 161–172.

A study of some particular frameworks in space.

17. A. L. Cauchy, (1905). *Sur les polygons et les polyèdres,* Oevres Complètes d'Augustin Cauchy 2è Série Tom **1**, 26–38.

A proof of: The 2-skeleton of a strictly convex polyhedron in 3-space is rigid.

18. R. Connelly, (1975). *An attack on rigidity I, II,* Bull. Amer. Math. Soc. **81**, 566–569.

A proof of: Any embedded suspension of a polygon circle is rigid in 3-space.

19. R. Connelly, (1976). *An immersed polyhedral surface which flexes,* Indiana Univ. Math. J. **25**, 965–972.

A review of Cayley's result and some examples.

20. R. Connelly, (1978). *A counter example to the rigidity conjecture for polyhedra,* Inst. Haut. Étud. Sci. Publ. Math. **47**, 333–335.

The first counter example to Euler's conjecture.

21. R. Connelly, (1978). *A flexible sphere,* Math. Intelligencer **1**, 130–131.

22. R. Connelly, (1979). *The rigidity of polyhedral surfaces,* Math. Mag. **52**, 275–283.

A short history of Euler's conjecture and a discussion of several counter examples.

23. R. Connelly, (1980). *The rigidity of certain cabled frameworks and the second-order rigidity of arbitrary triangulated convex surfaces,* Advances in Math. **37**, 272–298.

The concept of n'th-order rigidity is introduced to deal with frameworks that are rigid but not infinitesimally rigid, i.e. first order rigid. He proves that n'th order rigidity implies rigidity. He then extends Alexandrov's result proving that all convex polyhedra with triangular faces are rigid. Finally, he applies the theory of second-order rigidity to tensed frameworks.

24. R. Connelly, (1982). *Rigidity and energy,* Invent. Math. **66** 1, 11–33.

25. R. Connelly, (1990). *On generic global rigidity,* in "Applied Geometry and Discrete Mathematics", DIMACS Ser. Discrete Math. Theoret. Comput. Sci. **4**, 147–155.

$K_{5,5}$ is shown to be not generically globally (i.e. strongly) rigid in dimension 3, which disproves Hendrickson's conjecture that a graph is globally rigid in R^d if it is $(d+1)$-connected and generically redundantly rigid in R^d.

26. R. Connelly and H. Servatius, (1993). *Higher-order rigidity - What is the proper definition?* to appear in Discrete & Computational Geometry.
    Exhibits an example of a finite mechanism whose configuration space contains a cusp. The cusp point is a framework which turns out to be third order rigid, even though the framework is not rigid, calling into question the whole notion of higher order rigidity.
27. R. Connelly and W. Whiteley, (1990). *Second-order rigidity and pre-stress stability for tensegrity frameworks,* preprint.
    Shows that first-order rigidity implies pre-stress stability implies second-order rigidity implies rigidity, for any framework, and that none of these implications is reversible, even for bar frameworks.
28. H. Crapo, (1977). *More on bracing a one story building,* Environment and Planning B **4**, 153–156.
    Considers the bracing problem for planar grids in space supported by vertical rods of uniform length from a solid floor. (Incorporated into [**11**].)
29. H. Crapo, (1979). *Structural rigidity,* Structural Topology **1**, 26–45.
    This paper summarizes the then available general theory of rigidity in 2 and 3 space.
30. H. Crapo, (1982). *The tetrahedral/octahedral truss,* Structural Topology **7**, 52–60.
    He shows that this truss is not infinitesimally rigid.
31. H. Crapo, (1991). *Invariant-theoretic methods in scene analysis and structural mechanics,* Journal of Symbolic Computation **11** 5/6, 523–548.
    Contains the derivation of pure conditions via complementation (Hodge star) in the exterior algebra of an independent configuration. Remarks on Cayley factorization of projective conditions.
32. H. Crapo, (1993). *On the generic rigidity of structures in the plane,* Advances in Applied Math., to appear.
    Gives a new characterization of generic rigidity in the plane: A graph G is generically rigid in the plane iff it is the edge disjoint union of three connected subgraphs G_i of G such that every vertex of G is in exactly two of those subgraphs and no subgraph of G having at least one edge is spanned by subgraphs of two of the three subgraphs G_i. This characterization leads to an efficient algorithm to decide whether a given graph is generically rigid in the plane.

33. H. Crapo and J. Ryan, (1986). *Spatial realizations of linear scenes,* Structural Topology **13**, 33-68.
34. H. Crapo and W. Whiteley, (1982). *Statics of frameworks and motions of panel structures: a projective geometric introduction,* Structural Topology **6**, 42–82.

    Two traditionally separate fields, the statics of bar and joint frameworks and the instantaneous kinematics of articulated panel structures, are here united in a novel and appropriate projective geometry setting. Exterior algebra (or Cayley algebra) is employed to represent equilibrium loads and static stresses on a bar and joint framework, and to represent instantaneous motions of an articulated panel structure. In the special case of a framework and a panel structure, of a spherical polyhedron, a natural algebraic isomorphism between static stresses in the framework and instantaneous motions of the panel structure is exhibited. Under a suitable path condition on the stresses, this isomorphism is extended to frameworks and panel structures based on arbitrary oriented polyhedra.
35. H. Crapo and W. Whiteley, (1993). *Plane stresses and projected polyhedra,* Structural Topology, to appear.
36. H. Crapo and W. Whiteley, (1993). *Stressed frameworks and projected 4-polytopes,* to appear.
37. H. Crapo and W. Whiteley, (1993). *The Geometry of Rigid Structures,* Encyclopedia of Math., Cambridge University Press.
38. H. R. Cox, (1936). *On the synthesis and analysis of simply-stiff frameworks,* Proc. London Math. Soc. (2) **40**, 203–216.

    A study of generic structures in 2 and 3-space. A variety of results is proved concerning ways to add vertices to generically rigid frameworks or join rigid frameworks which result in larger generically rigid frameworks.
39. M. Dehn, (1916). *Über die Starrheit konvexer Polyeder,* Math. Ann. **77**, 466–473.
40. A. C. Dixon, (1899/1900). *On certain deformable frameworks,* Messenger Math. **29**, 1–21.

    Certain planar linkages are constructed and then used to prove a geometric theorem. Includes a study of all linkages with 1 degree of freedom, 4 pivots, 4 joints, and 8 rods.
41. J. Edmonds, (1965). *Minimum partition of a matroid into independent subsets,* J. Res. Nat. Bur. Standards B **69**, 67–72.

    Describes an efficient algorithm to decompose a given graph (matroid) into k disjoint trees (k bases) whenever possible.
42. N. V. Efimov, (1962). *Qualitative problems in the theory of deformable structures,* Translations Amer. Math. Series 1 **6**, 274–423.

43. L. Euler, (1862). *Opera Postuma I,* Petropoli, 494–496.

This is the paper in which Euler made his conjecture on the rigidity of closed polyhedral surfaces in 3-space.

44. A. Fogelsanger, (1988). *The generic rigidity of minimal cycles,* Ph.D. Thesis, Cornell University.

Proves that the 1-skeleton of any abstract triangulation of a 2-manifold is generically rigid in 3-space.

45. H. Gluck, (1975). *Almost all simply connected closed surfaces are rigid,* in "Geometric Topology", Springer Lecture Notes **438**, 225–239.

The author introduces both rigidity and infinitesimal rigidity and proves that the 1-skeleton of any triangulated simply connected closed surface in 3-space is generically rigid.

46. M. Goldberg, (1978). *Unstable polyhedral structures,* Math. Mag. **51**, 165–170.

Discusses several specific surfaces: rigid but not strongly rigid; rigid but not infinitesimally rigid; and Connelly's counter example.

47. Jack E. Graver, (1991). *Rigidity Matroids,* SIAM J. Discrete Math. **4**, 355-368.

Introduces the concept of an abstract rigidity matroid.

48. M. Grötschel, L. Lovász and A. Schrijver (1988). *Geometric Algorithms and Combinatorial Optimization,* Springer-Verlag, Berlin.

49. B. Grünbaum, (1978). *Lectures in lost mathematics,* Univ. of Washington.

Discusses the history of rigidity theory.

50. B. Grünbaum, (1988). *Rigid plate frameworks,* Structural Topology **14**, 1–8.

Studies plate frameworks in the plane in which every vertex intersects exactly two plates.

51. Open University, (1981). *Kinematics,* The Open University Press.

This text forms part of the correspondence element of an Open University Third Level Course. Investigates ways in which various rigid mechanical components can be interconnected so as to form a kinematic system of specific type. Kinematic structures are modeled by two different graph representations, the direct graph and the interchange graph. Emphasis is given to the combinatorial aspects of the subject.

52. B. Hendrickson, (1992). *Conditions for unique graph realizations,* SIAM J. Comput. **21**, 65–84.

Shows that flexes of generic structures are manifolds, and identifies generic redundant rigidity as a necessary condition for a generic structure to have a unique realization.

53. L. Henneberg, (1968). *Die graphische Statik der starren Systeme* Leipzig 1911, Johnson Reprint.

Summary of results of the German school of geometric statics from the last century, statics and mechanics of frameworks with replacement principles.

54. D. Hilbert and S. Cohn-Vossen, (1952). *Kinematics*, Chapter 5, in "Geometry and the Imagination", Cheslea, New York.
Discusses linkages in the plane and in 3-space.

55. M. Homma and H. Maehara, (1990). *Algebraic distance graphs and rigidity,* Trans. Amer. Math. Soc. **319** 2, 561–572.
Proves that an algebraic distance graph, i.e. a graph in which two vertices are adjacent if the distance between them is an algebraic number, is complete if an only if it is rigid.

56. A. W. Ingleton, (1971). *Conditions for representability and transversality of matroids,* Proc. Fr. Br. Conf. 1970, Springer Lecture Notes **211**, 62–67.

57. P. J. Kahn, (1979). *Counting types of rigid frameworks,* Invent. Math. **55**, 297–308.
This paper includes proofs of: For a fixed, graph there is a finite algorithm which will decide if a given embedding is rigid or not; For a fixed graph, the number of embeddings is finite up to *homotopic* embeddings.

58. G. Kalai, (1984). *Weakly saturated graphs are rigid,* Annals Discrete Math. **20**, 189-190.
A graph G of order n is weakly d-saturated if there is a sequence of graphs $G = G_0 \subseteq \ldots \subseteq G_t = K_n$ such that G_{i+1} has one edge more than G_i and the number of complete subgraphs of order d in G_{i+1} is strictly greater than that number in G_i. Shows that every general embedding of a weakly d-saturated graph $\mathbb{R}^{d-2}$ is rigid.

59. G. Kalai, (1985). *Hyperconnectivity of graphs,* Graphs and Combinatorics **1**, 65-79.
A new notion of connectivity is introduced by defining a matroid $\mathcal{M}_k^n$ on the edges of K_k via exterior algebra. A graph on n vertices is defined to be k-hyperconnected if its edge set spans $\mathcal{M}_k^n$. The connection between k-hyperconnectivity and rigidity is explored. For example, $\mathcal{M}_1^n$ is the connectivity matroid, and a 2-hyperconnected graph is generically rigid in the plane.

60. G. Kalai, (1990). *Symmetric matroids,* J. Comb. Theory B **50**, 54-64.
Considers symmetric matroids, including rigidity matroids, and their growth functions.

61. R. Kanayama, (1933). *Bibliography on the theory of linkages,* Tohoku Math. J. **37**, 294–319.
A bibliography of 306 papers on linkages published between 1631 and 1931.

62. A. B. Kempe, (1876). *On a general method of describing plane curves of the n'th degree by linkworks,* Proc. London Math. Soc. **7**, 213–216.

This paper contains a description of several basic linkages which in various configurations yield any curve defined by a polynomial in x and y set equal to zero.

63. A. B. Kempe, (1877). *How to draw a straight line,* London, Macmillan.

An elementary lecture on the theory and history of linkages.

64. G. Laman, (1970). *On Graphs and rigidity of plane skeletal structures,* J. Engrg. Math. **4**, 331–340.

This paper gives the first combinatorial characterization of generic rigidity in the plane.

65. H. Lamb, (1928). *Statics,* Cambridge Univ. Press.

66. H. A. Lauwerier, (1966). *A note on flexible hexagons,* Proc. Nether. Acad. Wet. **69**, 330–334.

Shows that a spatial hexagon the sides and angles of which are given is flexible if and only if its opposite elements are equal. (See also [**13**].)

67. H. Lebesgue, (1907). *Octaèdres articulés de Bricard,* Enseign. Math. (2) **13**, 175–185.

This paper discusses Bricard's flexible octahedron.

68. H. Lebesgue, (1909). *Démonstration complètes du théorem du Cauchy sur l'égalite des polyèdres convexes,* Intermédiaire Recherches Math. **16**, 113–120.

69. H. Liebmann, (1920). *Ausnahmefachwerke und ihre Determinante,* Sitzungsber. Bayer. Akad. Wiss. 187–227.

70. L. Lovász, (1979). *Combinatorial Problems and Exercises,* North Holland.

71. L. Lovász, (1977). *Flats in matroids and geometric graphs,* Combinatorial Surveys, Proc. 6th British Combinatorial Conf., P. Cameron, ed., Academic Press, New York, 45–69.

72. L. Lovász and A. Recski, (1982). *Selected topics of matroid theory and its applications,* Rendiconti del circolo Matematico di Palermo, II **2**, 171–185.

73. L. Lovász and Y. Yemini, (1982). *On Generic rigidity in the plane,* SIAM J. Alg. Disc. Methods **3**, 91–98.

This paper introduces infinitesimal and generic rigidity. It shows that generic rigidity is combinatorial. It contains a proof of *the cover theorem* for the planar generic case and uses this result to prove: A graph is generically independent if and only if doubling any edge results in a graph which is the union of two forests. Every 6-connected graph is generically rigid in the plane. With an example the last result is shown to be best possible.

74. L. A. Lyusternik, (1963). *Convex figures and polyhedra,* Dover Publications.

75. J. C. Maxwell, (1864). *On reciprocal figures and diagrams of forces,* Philos. Mag. **4** 27, 250–261.

also XXVI COLLECTED PAPERS, Cambridge Univ. Press (1890). Let (V, E) be a graph and let f be *the map* from real $m|V|$-space to real $|E|$-space which has $\mathcal{V}$, the space of infinitesimal motions of an embedding of (V, E), as kernel. Then the orthogonal complement of $\mathrm{im}(f)$ is the space of internal stresses of that embedding. It is not clear that Maxwell thought of things in this way. However, his study of internal stresses does give rigidity results because of this relation. Maxwell studies the space of stresses for planar frameworks by means of reciprocal figures. Given a framework, a second framework is reciprocal to the first if there exists a 1-1 correspondence between the edges such that: 1, corresponding lines are parallel; 2, the edges corresponding to a vertex bond in one figure form a circuit in the other figure. (See [**136**].)

76. J. C. Maxwell, (1869-1871). *On reciprocal figures, frames and diagrams of force,* Trans. Royal Soc. Edinburgh **26**, 1–40.

This paper is an extension of [**75**].

77. H. Maehara, (1988). *A simple proof for the maximal planar case of Steinitz's theorem,* Bull. Inst. Math. Acad. **16** 4, 373–376.

78. H. Maehara, (1989). *On the rigidity of complete multipartite graphs,* Ryukyu Math. J. **2**, 29–47.

79. H. Maehara, (1991). *On Graver's conjecture concerning the rigidity problem of graphs,* Discrete & Computational Geometry **6** 4, 339–342.

This paper shows that a 2-extension of a 4-isostatic set is not necessarily 4-isostatic. The counterexample presented here may easily be extended to higher dimensions.

80. H. Maehara, (1991). *A rigid unit-bar-framework without triangles,* Mathematica Japonica, **36** 4, 681–683.

An elementary construction of a rigid triangle-free bar and joint framework in the plane in which all bars have equal length.

81. H. Maehara, (1991). *Distances in a rigid unit-distance graph in the plane,* Discrete. Applied Math. **31** 2, 193–200.

82. H. Maehara, (1992). *Extending a flexible unit-bar framework to a rigid one,* Discrete Math. **108** 1/3, 167–172.

83. E. W. Parks, (1965). *Braced frameworks,* Pergammon Press, London.

84. A. V. Pogorelor, (1973). *Extrinsic geometry of convex surfaces,* Transl. Math. Monographs 35, Amer. Math. Soc.

Alexandrov's work is presented here.

85. A. Pugh, (1976). *An introduction to tensegrity,* Univ. of Calif. Press.

Some architectural examples are discussed here.

86. A. Recski, (1982). *Engineering applications of matroids - a survey,* A. Barlotti, (ed) Matroid Theory and its Applications, Liguori editore, Napoli, 299–321.
87. A. Recski, (1984). *A network theory approach to the rigidity of skeletal structures. Part 1. Modeling and interconnection,* Discrete Applied Math. **7**, 313–324.
88. A. Recski, (1984). *Statics and electrical network theory: A unifying role of matroids,* W. R. Pulleyblank (ed), Progress in Combinatorial Optimization, Academic Press, London, 307–314.
89. A. Recski, (1984). *A network theory approach to the rigidity of skeletal structures. Part 2. Laman's Theorem and topological formulae,* Discrete Applied Math. **8**, 63–68.
90. A. Recski, (1984). *Applications of combinatorics to statics - a survey,* Rendiconti del circolo Matematico di Palermo, II, **3**, 237–247.
91. A. Recski, (1984). *A network theory approach to the rigidity of skeletal structures. Part 3. Electric model of planar frameworks,* Structural Topology **9**, 59–71.
92. A. Recski, (1988/89). *Bracing cubic grids - a necessary condition,* Discrete Math. **73**, 199–206.
93. A. Recski, (1989). *Matroid Theory and its Applications in Electrical Network Theory and in Statics,* Springer Verlag, Berlin.
94. A. Recski, (1989). *Symmetric bracing of one-story buildings with cables and asymmetric bracings of one-story buildings with rods,* Symmetry of Structures, Budapest, **II**, 471–472.
95. A. Recski, (1991). *One Story buildings as tensegrity frameworks II,* Structural Topology **17**, 43–52.
96. A. Recski, (1991). *Applications of combinatorics in structural engineering,* in "First Workshop on Applications of Combinatorial Optimization in Science and Technology", Rutgers University, New Brunswick NJ, 275–282.
97. A. Recski, (1992). *Applications of combinatorics to statics–A second survey,* Discrete Math. **108** 1/3, 183–188.
98. A. Recski, (1993). *Combinatorics in Electrical Engineering and Statics,* Handbook in Combinatorics, R. Graham, M. Grötschel, and L. Lovász (eds.) North Holland, Amsterdam.
99. A. Recski and W. Schwarzler, (1992). *One-story buildings as tensegrity frameworks III,* Discrete Applied Math. **39**, 137–146.
100. I. G. Rosenberg, (1979). *Structural rigidity of bar & joint and cabled frameworks,* Proc. 10th S-E Conf. Combinatorics, Graph Theory and Computing, 843–848.

    The rigidity of frameworks is studied in terms of an infinite system of equations obtained from power series expansions of trajectories. The system is used for deriving sufficient conditions for rigidity going beyond the standard infinitesimal rigidity.

101. I. G. Rosenberg, (1980). *Structural rigidity I,* Annals Discrete Math. **8**, 143–161.

Structural rigidity of tensegrity (i.e. bar, cable, and strut) frameworks is studied. The frameworks are fixed in space so that no euclidean rigid motions may occur. A rigidity criterion reduces the problem to the check whether an infinite system of equations has trivial solutions only. The paper concludes with the infinitesimal rigidity of tensegrity frameworks.

102. I. G. Rosenberg, (1986). *Structural rigidity II,* Discrete Applied Math. **13**, 41–59.

Structural rigidity of bar frameworks is studied in the case that the corresponding rigidity matrix has rank one less than full. A rigidity criterion for these almost infinitesimally rigid frameworks is derived from a similar criterion in [**101**].

103. B. Roth, (1981). *Rigid and flexible frameworks,* Amer. Math. Monthly **88**, 6–20.

An expository paper on rigidity.

104. B. Roth and W. Whiteley, (1981). *Tensegrity frameworks,* Trans. Amer. Math. Soc. **256**, 419–445.

This paper establishes some basic results concerning the rigidity, flexibility, infinitesimal rigidity and infinitesimal flexibility of tensegrity frameworks. These results are then applied to a number of questions, problems and conjectures regarding tensegrity frameworks in the plane and in space.

105. H. J. Ryser, (1963). *Combinatorial Mathematics,* Carus Mathematical Monographs 14, AMS.

106. D. Sachs, (1970). *Graphs, matroids, and geometric lattices,* J. Comb. Theory **9**, 192–199.

107. I. Schoenberg and S. K. Zaremba, (1967). *On Cauchy's lemma concerning convex polygons,* Canadian J. Math. **19**, 1062-1071.

This paper contains several correct proofs of a lemma used by Cauchy in proving his rigidity theorem.

108. B. Servatius, (1989). *Birigidity in the plane,* SIAM J. Discrete Math. **2** 4, 582–589.

Defines edge and vertex birigidity, and investigates the relationship between the connectivity of the rigidity matroid and and the birigidity of the underlying graph.

109. E. Steinitz and H. Rademacher, (1934). *Vorlesungen über die Theorie der Polyeder,* Springer Verlag.

This book contains a correct proof (the first?) of Cauchy's Rigidity Theorem.

110. J. J. Stoker Jr., (1968). *Geometrical problems concerning polyhedra in the large,* Comm. Pure Applied Math. **21**, 119–168.

The purpose of this paper is to prove and at the same time generalize Cauchy's Rigidity Theorem and to extend his basic ideas to a number of other cases, including polyhedra with boundaries, certain non-convex polyhedra, and special examples of polyhedra of arbitrary genus.

111. K. Sugihara, (1980). *On redundant bracing in plane skeletal structures,* Bulletin of the Electrotechnical Laboratory **44** 5/6, 78–88.

    Develops the idea of cycles in the 2-dimensional generic rigidity matroid and provides several nice examples.

112. T. Tarnai and E. Makai, (1989). *A movable pair of tetrahedra,* Proc. Royal Soc. London Ser. A **423** 1865, 419–442.

    Analyses the example of two regular tetrahedra in dual position constrained to move such that the dual edges intersect.

113. T. S. Tay, (1976). *Circuits in three dimensional structural geometry,* M. Math. Thesis, Univ. Waterloo.

114. T. S. Tay, (1981). *Rigidity problems in bar and joint frameworks,* Ph.D. Math. Thesis, Univ. Waterloo.

    By a rigid body in n-space, the author means an n-dimensional solid. An abstract linkage is a multigraph and a realization of a linkage in n-space consists of embedding the multigraph in n-space and replacing the vertices by rigid bodies. The main result is: an abstract linkage has a rigid realization in n-space if and only if it contains $n(n+1)/2$ edge disjoint spanning trees.

115. T. S. Tay, (1984). *Rigidity of multi-graphs I: Linking rigid bodies in n–space,* J. Comb. Theory B. **36**, 95–112.

116. T. S. Tay, (1984). *Rigidity of multigraphs II,* Graph Theory Singapore, Springer Lect. Notes in Math **1073**, 129–134.

    This paper gives a short proof of the characterization theorem of an n–linkage. These structures are now known as bar and body frameworks in n–space.

117. T. S. Tay, (1989). *Linking $(n-2)$-Dimensional Panels in n-Space II: $(n-2,2)$-Frameworks and Body and Hinge Structures,* Graphs and Combinatorics **5** 3, 245–273.

118. T. S. Tay, (1991). *Linking $(n-2)$-Dimensional Panels in n-Space I: $(k-1,k)$-Graphs and $(k-1,k)$-Frames,* Graphs and combinatorics **7** 3, 289–304.

119. T. S. Tay, (1991). *Henneberg's method for bar and body frameworks,* Structural Topology **17**, 53–58.

    The main result is an inductive construction of generic isostatic bar and body frameworks, namely frameworks of rigid bodies linked by bars.

120. T. S. Tay, (1993). *On Generically dependent bar and joint frameworks in space,* To appear in Structural Topology.

121. T. S. Tay, (1993). *On the infinitesimal rigidity of an n–ring,* Acta Mechanica, to appear.

An n–ring in 3–space consists of an n–gon in space supported at each vertex by two bars attached to the ground. This paper gives a characterization of infinitesimally rigid n–rings.

122. T. S. Tay, (1993) *A new proof of Laman's theorem,* preprint.

Gives a direct proof of Crapo's characterization of generic isostatic bar frameworks in the plane as the union of 3 trees such that every vertex is covered by exactly 2 of them.

123. T. S. Tay, N. L. White and W. Whiteley, (1993). *Skeletal rigidity of simplicial complexes I,* preprint.

124. T. S. Tay, N. L. White and W. Whiteley, (1993). *Skeletal rigidity of simplicial complexes II,* preprint,

The central theme of the these two papers is a generalized concept of rigidity, namely skeletal rigidity of simplicial complexes. Five different but equivalent rigidity matrices are defined. The relationship between the g–theorem of convex polytopes and skeletal rigidity is also established.

125. T. S. Tay and W. Whiteley, (1984). *Recent advances in generic rigidity of structures,* Structural Topology **9**, 31–38.

126. T. S. Tay and W. Whiteley, (1985). *Generating isostatic frameworks,* Structural Topology **11**, 21–69.

This paper contains several conjectures and a useful bibliography.

127. S. Timoshenko and D. H. Youngs, (1945). *Theory Of Structures,* McGraw-Hill, New York.

128. N. J. Thurston, (1991). *On the rigidity of graphs,* Bachelor of Arts Thesis, Reed College.

Shows that the maximal conjecture fails in dimension four.

129. W. P. Thurston and J. R. Weeks, (1984). *The mathematics of three–dimensional manifolds,* Scientific American **251** 1, 108–120.

A readable introduction to the geometry of manifolds in 3 and higher dimensions. Configuration spaces of frameworks are used as examples.

130. W. T. Tutte, (1958). *A homotopy theorem for matroids I and II,* Trans. Amer. Math. Soc. **88**, 144–174.

131. W. T. Tutte, (1965). *Lectures on Matroids,* J. Res. Nat. Bur. Stand. **69B**, 1–48.

132. D. J. A. Welsh, (1976). *Matroid Theory,* Academic Press, London.

133. N. White and W. Whiteley, (1983). *The algebraic geometry of stress in frameworks,* SIAM J. Algebraic Discrete Methods **4**, 481–511.

This paper studies the algebraic - geometric conditions called the pure condition for infinitesimal rigidity or for stress on a bar and joint framework.

134. N. White and W. Whiteley, (1987). *The algebraic geometry of motions of bar and body frameworks,* SIAM J. Algebraic Discrete Methods **8**, 1–32.

A study of the algebraic-geometric conditions for infinitesimal rigidity, and a method for determining the motions of a bar and body framework with 1 degree of freedom.

135. W. Whiteley, (1979). *Motions of bipartite frameworks,* Structural Topology **3**, 62–63.

Some examples of the result obtained in [**12**] are given.

136. W. Whiteley, (1982). *Motions and stresses of projected polyhedra,* Structural Topology **7**, 13–38.

Using infinitesimal motions of panel structures, a new proof is given for Maxwell's theorem that the projection of an oriented polyhedron from 3-space gives a plane diagram of lines and points which forms a stressed bar and joint framework. The methods extend to prove a simple converse and a general converse for other polyhedra under appropriate conditions on the stress.

137. W. Whiteley, (1982). *Motions of trusses and bipartite frameworks,* Structural Topology **7**, 61–68.

A continuation of the work of Crapo, [**30**]. The same truss is studied after some of the symmetry is destroyed.

138. W. Whiteley, (1983). *Cones, infinity and one story buildings,* Structural Topology **8**, 53–70.

Consider a one story building constructed with a series of vertical columns (possibly of different lengths), a roof framework connecting the tops of the columns, plus a minimum of three additional wall braces going to the tops of some columns. We examine the static rigidity (i.e. infinitesimal rigidity) of this framework by viewing it as a cone from the point at infinity.

139. W. Whiteley, (1984). *Infinitesimal motions of a bipartite framework,* Pacific J. Math. **110**, 233–255.

A recent established criterion [**12**] for the stresses on any bar and joint framework with a complete bipartite graph is converted into explicit criteria for infinitesimal motions of the framework.

140. W. Whiteley, (1984). *Infinitesimally rigid polyhedra I: statics of frameworks,* Trans. Amer. Math. Soc. **285** 1, 431–461.

A proof of Alexandrov's Theorem using *statics*; also an extension of this result to 4-space.

141. W. Whiteley, (1987). *Rigidity and polarity I: Statics of sheet structures,* Geometriae Dedicata **22** 3, 329–362.

Relates the rigidity of polarized frameworks in projective space to euclidean sheet structures.

142. W. Whiteley, (1988). *The union of matroids and the rigidity of frameworks,* SIAM. J. Discrete Math. **1** 237–255.

Shows an m-dimensional infinitesimal rigidity matroid contains m-copies of the cycle matroid.

143. W. Whiteley, (1988). *Infinitesimally rigid polyhedra II: Modified spherical frameworks,* Trans. Amer. Math. Soc. **306** 1, 115–139.

Replacement techniques are developed and used to prove that an inseparable triangulated sphere, with one edge removed, flexes so that all dihedral angles change, resolving a conjecture of Kuiper.

144. W. Whiteley, (1989). *A matroid on hypergraphs, with applications in scene analysis and geometry,* Discrete Comput. Geom. **4** 1, 75–95.

Resolves a conjecture by Sugihara that the matroid defined for liftings of 'polyhedral configuations' of plane points and some collections of points designated faces, to spatial configurations with plane faces is characterized by a simple submodular function on the bipartite incidence graph. This result is extended to study liftings between adjacent dimesnions, and a new proof of Laman's Theorem results from the study of liftings from the line to the plane. Connections are made to the dual parallel redrawing matroid.

145. W. Whiteley, (1989). *Rigidity and polarity II: Weaving lines and tensegrity frameworks,* Geometriae Dedicata **30** 3, 255-279.

A weaving is a pattern of straight lines in the plane with specified over and under crossings marked at each intersection. The question of which weavings arise from lines embedded in space is related to tensegrity frameworks.

146. W. Whiteley, (1990). *Vertex splittings in isostatic frameworks,* Structural Topology **16**, 23–30.

Shows how to extend an independent or rigid set by splitting a vertex.

147. W. Whiteley, (1991). *Weavings, sections and projections of spherical polyhedra,* Discrete Applied Math. **32** 3, 275-294.

A proof that the projection of a polyhedron and a cross section of the same polyhedron are 'reciprocal' – a self stress in the projection corresponds to a weaving pattern in the cross section. A form of the theory is invariant under plane polarity.

148. W. Whiteley, (1991). *The combinatorics of bivariate splines,* in "Applied Geometry and Discrete Mathematics", DIMACS Ser. Discrete Math. Theor. Comp. Sci. **4**, 587-608.

A description of the generic matroid for piecewise quadratic, globally C^1 functions over a dissected polygon in the plane. Includes a proof of a series of properties reflecting a strong analogy to the generic rigidity in 3-space of the graph of the dissection if the boundary polygon is a triangle. Analogous properties include: rank $3|V| - 6$ on complete graphs of more than one vertex; vertex splitting; the independence of triangulated spheres; etc. Some discussion of related matroids analogous to rigidity in higher dimensions, but defined for realizations of the graph in the plane. (Note that, by Maxwell's Theorem, piecewise linear, globally continuous functions are analogous to problems of plane stresses).

149. W. Whiteley, (1992). *Matroids and rigid structures,* in "Matroid Applications", Neil White, ed. Encyclopedia of Mathematics and its Applications **40**, Cambridge University Press, 1–53.

    A summary of connections between matroids and rigidity, as well as related concepts such as polyhedral pictures and bivariate splines.

150. H. Whitney, (1932). *Congruent graphs and the connectivity of graphs,* Amer. J. Math. **54**, 150–168.
151. H. Whitney, (1935). *On the abstract properties of linear independence,* Amer. J. Math. **57**, 509–533.
152. W. Wunderlich, (1954). *Ein merkwürdiges Zwölfstabgetriebe,* Österr. Ingen. Archiv. **8**, 224–228.
153. W. Wunderlich, (1965). *Starre, kippende, wackelige und bewegliche Achtflache,* Elem. Math. **20**, 25–32.
154. W. Wunderlich, (1971). *Starre, kippende, wackelige und bewegliche Gelenkvierecke im Raum,* Elem. Math. **26**, 73–83.
155. W. Wunderlich, (1976). *On deformable nine bar linkages with six triple joints,* Proc. K. Nederl. Akad. Wetensch Proc. A. **79**, 255–262.
156. W. Wunderlich, (1977). *Gefährliche Annahmen der Trilateration und bewegliche Fachwerke I, II,* Z. Angnew. Math. Mech. **57**, 297–304, 363–367.
157. W. Wunderlich, (1978). *Sur une déformation remarquable du systèm des génératrices d'un cône du second degrè et un problème de géodésie,* Bul. Inst. Polit. Iasi. **24**, 81–85.
158. W. Wunderlich, (1979). *Snapping and shaky antiprisms,* Math. Magaz. **52**, 235–236.
159. W. Wunderlich, (1980). *Neue Wackelikosaeder,* Anz. Österr. Akad. Wiss. **117**, 28–33.
160. W. Wunderlich, (1980). *Wackelige Doppelpyramiden,* Anz. Österr. Akad. Wiss. **117**, 82–87.
161. W. Wunderlich, (1980). *Zur projectiven Invarianz von Wackelstrukturen,* Z. Angnew. Math. Mech. **60**, 703–708.

162. W. Wunderlich, (1981). *Wackelikosaeder,* Geometriae Dedicata **11**, 137–146.

163. W. Wunderlich, (1982). *Projective invariance of shaky structures,* Acta Mechanica **42**, 171–181.

164. W. Wunderlich, (1982). *Ringartige Wackelpolyeder,* Anz. Österr. Akad. Wiss. **119**, 71–77.

165. W. Wunderlich, (1982). *Wackeldodekaeder,* Elem. Math. **37**, 153–163.

166. W. Wunderlich, (1983). *Über Ausnahmefachewerke, deren Knoten auf einem Kegelschnitt liegen,* Acta Mechanica **47**, 153–163.

167. W. Wunderlich and C. Schwabe, (1986). *Eine Familie von geschlossenen gleichflächigen Polyedern, die fast beweglich sind,* Elem. Math. **41**, 88–98.

168. W. Wunderlich, (1990). *Shaky polyhedra of higher connection,* Publ. Math. Debrecen **37**, 355–361.

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