The Integrals of Lebesgue, Denjoy, Perron, and Henstock

Russell A. Gordon

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ABSTRACT. Suppose that $f : [a, b] \to R$ is differentiable at each point of $[a, b]$. Is f' integrable on $[a, b]$? The answer to this question depends on the integral that is used. For example, the answer is no for the Riemann and Lebesgue integrals. In this century, three integration processes have been developed that provide an affirmative answer to this question. The principal investigators of these integrals were Denjoy, Perron, and Henstock. Each of these integrals generalizes a different property of the Lebesgue integral, but it turns out that all three integrals are equivalent.

In this book, the properties of the Lebesgue, Denjoy, Perron, and Henstock integrals are developed fully from their definitions. The equivalence of the last three integrals is then established. Discussions of the integration by parts formula and convergence theorems are included. In the last part of the book, we consider approximate derivatives and attempts to develop an integration process for which every approximate derivative is integrable.

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To Brenda and Charles

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Preface

The Fundamental Theorem of Calculus is one of the highlights of a first year calculus course. Its importance lies in the fact that it reveals a significant relationship between integration and differentiation. As usually stated in calculus books, the functions involved are continuous. However, the following version of the Fundamental Theorem of Calculus is valid.

> If F is differentiable on $[a, b]$ and if F' is Riemann integrable on $[a, b]$, then $\int_a^x F' = F(x) - F(a)$ for each $x \in [a, b]$.

At first glance, it might appear that there is an extra hypothesis. Aren't all derivatives Riemann integrable? A brief search leads one to derivatives that are not bounded and, as a result, not Riemann integrable. However, there are even bounded derivatives (existing at all points) that are not Riemann integrable. In other words, the extra hypothesis is essential.

What about the Lebesgue integral? This integral was designed to overcome the deficiencies of the Riemann integral. Are all derivatives Lebesgue integrable? The answer once again is no. However, all bounded derivatives are Lebesgue integrable so the following version of the Fundamental Theorem of Calculus is valid.

> If F is differentiable on $[a, b]$ and if F' is bounded on $[a, b]$, then F' is Lebesgue integrable on $[a, b]$ and $\int_a^x F' = F(x) - F(a)$ for each $x \in [a, b]$.

This discussion leads naturally to the following question. Is it possible to define an integration process for which the theorem

> If F is differentiable on $[a, b]$, then the function F' is integrable on $[a, b]$ and $\int_a^x F' = F(x) - F(a)$ for each $x \in [a, b]$.

is valid? The answer is yes. In this century, three integration processes have been developed for which this ideal version of the Fundamental Theorem of Calculus is valid. These integrals, named after their principal investigators Denjoy, Perron, and Henstock, each generalize some aspect of the Lebesgue integral. Since each of these new integrals focuses on a different property of the Lebesgue integral, the definitions of the integrals are radically different. However, it turns out that all three integrals are equivalent. The purpose of this book is to present, in an elementary fashion, these integration processes.

The material for this book has been drawn from a number of different sources. The books by Saks and Natanson are the standard references for the Denjoy and Perron integrals. However, Natanson presents little more than a brief introduction to these integrals and the classic work by Saks is not easily accessible to a novice in the field. The theory of the Henstock integral and recent developments related to all three integrals exist primarily in research articles. For this reason, it is difficult to follow these new developments. In addition, there have been some dead ends, the notation is inconsistent and often confusing, and some of the proofs contain errors. A detailed treatment of all three integrals from the ground up does not exist. As these integrals have found several applications and since they are interesting in their own right, it seems fitting that such a development should be available. The hope here is that a reader with a thorough understanding of basic real analysis, such as that found in Rudin's *Principles of Mathematical Analysis*, will be able to read this book.

The book essentially has three parts. Chapters 1 through 4 provide an introduction to the Lebesgue integral. No familiarity with the Lebesgue integral or the concept of measure is assumed. However, it is not the purpose of this book to provide an in-depth study of the Lebesgue integral. Only the bare essentials of Lebesgue integration are discussed here, but these essentials are discussed in full detail. The driving force is to present those aspects of the Lebesgue integral that are necessary to develop the Denjoy, Perron, and Henstock integrals. These first four chapters have been used successfully as an independent study course for advanced undergraduates.

The next section of the book, Chapters 7 through 13, presents the integrals of Denjoy, Perron, and Henstock. The definition and basic properties of each integral is considered in a separate chapter. Included in each of these chapters is, of course, a proof that the integral satisfies the ideal form of the Fundamental Theorem of Calculus. Chapter 11 establishes the equivalence of these three integrals. Chapter 10 is an aside to consider the McShane integral, a Riemann type integral that is equivalent to the Lebesgue integral. The integration by parts formula for each of these integrals is the primary focus of Chapter 12. An equally important part of this chapter is a brief introduction to the Riemann-Stieltjes integral. Convergence theorems are the topic of Chapter 13. A general convergence theorem for each integral is stated and proved, then it is shown which of these theorems is the most general. Some essential prerequisites for this section of the book are discussed in Chapters 5 and 6. These chapters form a bridge between the Lebesgue integral and its generalizations. The level of sophistication is increased at this point. The concepts of Darboux function, Baire class one function, and functions of generalized bounded variation are defined and studied.

Another transition occurs in Chapter 14. At this point, the notions of approximate continuity and approximate derivative are introduced. It is shown how these concepts fit naturally in the theory of the Lebesgue integral. A number of interesting facts about the approximate derivative are stated and proved. The last part of the book, Chapters 15 through 17, considers integration processes that attempt to recover a function from its approximate derivative. In other words, the focus changes to the following theorem.

> If F is approximately differentiable on $[a, b]$, then F'_{ap} is integrable on $[a, b]$ and $\int_a^x F'_{\text{ap}} = F(x) - F(a)$ for each $x \in [a, b]$.

This section looks at the Khintchine integral as well as some recent attempts to generalize the Henstock and Perron integrals to the case in which the indefinite integral is only approximately continuous. The book ends with some open questions related to these integrals.

Each of the chapters contains a set of exercises. Some of the exercises are simple consequences of definitions and theorems. These are intended to provide some practice with the concepts. Other exercises introduce important/interesting results that are not discussed in the text. It should be pointed out that some of the exercises are rather difficult. The order of the exercises corresponds to the order of the text. It may happen that a problem is trivial as a result of a later theorem. In such a case, it is usually intended that the problem be solved directly from the definition or earlier results. The reader who desires to develop a working knowledge of this subject should try all of the exercises. Complete solutions to all of the exercises are provided.

Most, but not all, of the notation is standard. A notation index can be found at the back of the book. A short list of references is also given. This list is by no means exhaustive. Several of these works (especially the book by Bruckner and the paper by Bullen on nonabsolute integrals) have extended lists of articles in them and should be consulted by the reader who wants to dig deeper. Another good source is the journal *Real Analysis Exchange*. There has been no attempt in this work to trace the history of the subject and, when simpler, newer proofs are given rather than those found in the original sources. As mentioned above, this work is drawn from a number of different sources and the author is deeply indebted to them. However, my own style and methods are evident throughout. Some of the proofs and ideas in this book are entirely new. I have worked independently on this book and would appreciate any comments, suggestions, and/or questions. Correspondence should be sent to the author at Whitman College, Walla Walla, WA 99362. A response will be sent as quickly as possible.

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Notation Index

Z^+	set of positive integers
Q	set of rational numbers
R	set of real numbers
R_e	extended real numbers, $R \cup \{\pm\infty\}$
$\overline{E}$	closure of the set E
E°	interior of the set E
$\mathcal{C}E$	complement of the set E
∂E	boundary of the set E
E^d	set of points of density of E that belong to E
χ_E	characteristic function of the set E
$\ell(I)$	length of interval I
$\mu(E)$	measure of the set E
$\mu^*(E)$	outer measure of the set E
$\mu_*(E)$	inner measure of the set E
$d_x E$	density of the set E at the point x
$\rho(x, E)$	distance from the point x to the set E
$d(A, B)$	distance between the sets A and B
$A \subseteq B$	A is a subset of B or $A = B$
$A - B$	$A \cap \mathcal{C}B$
$A \triangle B$	symmetric difference of the sets A and B
G_δ set	any set that is a countable intersection of open sets
F_σ set	any set that is a countable union of closed sets
$f(x+)$	$\lim\limits_{t \to x^+} f(t)$
$f(x-)$	$\lim\limits_{t \to x^-} f(t)$
f^+	$f^+(x) = \max\{f(x), 0\}$
f^-	$f^-(x) = \max\{-f(x), 0\}$
m_f	$m_f(x) = \lim\limits_{r \to 0^+} \inf\{f(t) : t \in (x - r, x + r) \cap [a, b]\}$
M_f	$M_f(x) = \lim\limits_{r \to 0^+} \sup\{f(t) : t \in (x - r, x + r) \cap [a, b]\}$

ϕ_f	distribution function of f
$F'(x)$	derivative of F at x
$F'_{\text{ap}}(x)$	approximate derivative of F at x
$F'_{\text{st}}(x)$	strong derivative of F at x
$\overline{D}F(x)$	upper derivate of F at x
$\underline{AD}F(x)$	lower approximate derivate of F at x
$D_+F(x)$	lower right derivate of F at x
$AD^-F(x)$	upper left approximate derivate of F at x
$\overline{SD}F(x)$	upper strong derivate of F at x
$\omega(F, [a, b])$	oscillation of the function F on the interval $[a, b]$
$V(F, [a, b])$	variation of the function F on the interval $[a, b]$
$V(F, E)$	weak variation of the function F on the set E
$V_*(F, E)$	strong variation of the function F on the set E
$\text{sgn } x$	1 if $x > 0$, 0 if $x = 0$, and -1 if $x < 0$
$\mathcal{B}$	collection of Borel sets
$\mathcal{P}$	a collection of non-overlapping tagged intervals
$\mathcal{D}$	a collection of non-overlapping free tagged intervals
Δ	an approximate full cover
Δ_E	approximate full cover with tags in the set E
U_a^b	$U(b) - U(a)$
$f(\mathcal{P})$	Riemann sum associated with $\mathcal{P}$
$f(\mathcal{P}^\phi)$	Riemann-Stieltjes sum associated with ϕ and $\mathcal{P}$
$F(\mathcal{P})$	sum of the increments of F over $\mathcal{P}$

Subject Index

This book provides an elementary, self-contained presentation of the integration processes developed by Lebesgue, Denjoy, Perron, and Henstock. The Lebesgue integral and its essential properties are first developed in detail. The other three integrals are all generalizations of the Lebesgue integral that satisfy the ideal version of the Fundamental Theorem of Calculus for differentiable functions on an interval. One of the book's unique features is that the Denjoy, Perron, and Henstock integrals are each developed fully and carefully from their corresponding definitions. The last part of the book is devoted to integration processes which satisfy a theorem analogous to the Fundamental Theorem for approximately differentiable functions. This part of the book is preceded by a detailed study of the approximate derivative and ends with some open questions. This book contains over 230 exercises (with solutions) that illustrate and expand the material in the text.

This is an excellent textbook for graduate students who have backgrounds in real analysis.

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