# Discovering Modern Set Theory. I
## The Basics

**Winfried Just**
**Martin Weese**

**Graduate Studies in Mathematics**
**Volume 8**

**American Mathematical Society**

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**Editorial Board**

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ABSTRACT. This volume is an introduction to set theory for beginning graduate students. It covers the basics of set theory that are considered prerequisites for other areas of mathematics such as ordinals, cardinals, transfinite induction and recursion, and applications of Zorn's lemma. It also contains a description of how mathematics can be founded on axiomatic set theory, as well as a discussion of the nature of consistency results.

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*To Ewelina and Veronika*

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# Contents

# Preface

This text grew out of lecture notes written[1] by Martin Weese of Humboldt Universität, Berlin, Germany in 1992/93. Winfried Just used part of these notes during the same academic year in a lecture course on set theory for first and second year graduate students at Ohio University, Athens, Ohio. While doing this, the idea of writing up an English text of the lecture crossed his mind. The idea proved irresistible enough to result in this book.

The genesis of this text accounts for some departures from the traditional textbook format. Generally speaking, this is not so much a textbook that could form the backbone of a lecture, but rather the text of the lecture itself. Accordingly, our language is perhaps closer to spoken, colloquial English than to the standard style of mathematical writing. Frequently, the text takes on the form of a dialogue between the authors and the reader. The exercises form an integral part of this dialogue and are not relegated to the end of sections.

We tried to keep the length of the text moderate. This may explain the absence of many a worthy theorem from this book. Our most important criterion for inclusion of an item was frequency of use outside of pure set theory. We want to emphasize that "item" may mean either an important concept (like "equiconsistency with the existence of a measurable cardinal"), a theorem (like Ramsey's Theorem), or a proof technique (like the craft of using Martin's Axiom). Therefore, we occasionally illustrate a technique by proving a somewhat marginal theorem. Of course, the "frequency of use outside set theory" is based on our subjective perceptions.

We do feel some remorse for the total exclusion of descriptive set theory from this text. This is a very important branch of set theory, and it overlaps with several other areas of mathematics, most notably topology and recursion theory. However, just adding a section on descriptive set theory to a text like this would look artificial and do no justice to the area. It seems to us that one should either cover a lot of descriptive set theory, or none at all.

At the end of most sections, there are "Mathographical Remarks." Their purpose is to show where the material fits in the history and literature of the subject. We hope they will provide some guidance for further reading in set theory. They should not be mistaken for "scholarly remarks" though. We did not make any effort whatsoever to trace the theorems of this book to their origins. However, each of the theorems presented here can also be found in at least one of the more specialized texts reviewed in the "Mathographical Remarks." Therefore, we do not feel guilty of severing chains of historical evidence.

This book owes its existence as much to our students and colleagues as it does to its authors. Parts of the original version of Martin Weese's lecture notes were

[1] In German.

read by students of Humboldt University, Berlin. We thank them for pointing out mistakes and suggesting improvements.

We are also indebted to Jörg Brendle for many thoughtful comments on the German version.

Special thanks are due to Mary Anne Swardson of Ohio University who read the very first English version of this text and generously applied her red pencil to it.[2] We are much indebted to her for this invaluable service.

We thank Howard Wicke of Ohio University, Marion Scheepers of Boise State University, and Frank Tall of the University of Toronto for reading parts of later versions and commenting on their shortcomings. Last but not least, we thank Ohio University students Brian Johnson, Mark McKibben, Todd Allin Morman, William Stamp, and Mark Starr for struggling through parts of this book and making many valuable suggestions for improvement.

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[2]As the old saying goes: "Spare the red and spoil the text."

# How to Read this Book

*So we shall now explain how to read the book. The right way is to put it on your desk in the day, below your pillow at night, devoting yourself to the reading, and solving the exercises till you know it by heart. Unfortunately, I suspect the reader is looking for advice how not to read, i.e. what to skip, and even better, how to read only some isolated highlights.*

(Saharon Shelah in the introduction to his book "Classification Theory and the Number of Non-Isomorphic Models")

*In mathematics, as anywhere today, it is becoming more difficult to tell the truth. To be sure, our store of accurate facts is more plentiful now than it has ever been. ... Unfortunately, telling the truth is not quite the same as reciting a rosary of facts.*

(Gian-Carlo Rota, 1985)

*W. A. Hurwitz used to say that in teaching on an elementary level one must tell the truth, nothing but the truth, but not the* whole *truth.*

(Mark Kac, 1976)

We wrote this text for two kinds of readers: beginning graduate students who want to get some grounding in set theory, and more advanced mathematicians who wish to broaden their knowledge of set theory. Furthermore, we wanted this text to be useful both as a textbook for a regular graduate course, as well as for those readers who wish to use it without the guidance of an instructor.

Volume I contains the basics of modern set theory. Many graduate texts on analysis, algebra, topology, or measure theory begin with a review of parts of this material as "set-theoretic prerequisites." Thus, Volume I is primarily aimed at beginning graduate or advanced undergraduate students. It can be used as a textbook in an introductory set theory course, or as supplementary reading in a course that relies heavily on set-theoretic prerequisites. Volume II is aimed at more advanced graduate students and research mathematicians specializing in fields other than set theory. It contains short but rigorous introductions to various set-theoretic techniques that have found applications outside of set theory. Although we think of Volume II as a natural continuation of Volume I, each volume is sufficiently self-contained to be studied separately. Since our terminology is fairly standard, more advanced students may be able to skip the first few sections of Volume I or even go directly to Volume II.

If you do not have the benefit of an instructor who can tell you what to skip, the best policy is to proceed as follows: Read the Introduction. It will give you some general idea what we are up to. If you don't understand every word of it, don't worry. Next, find out at which point of the rest of the book things start to look new to you, and begin reading right there. If things start to look new only around Chapter 22 or so, you probably do not want to waste your time reading this book, but go to the more advanced literature on the subject. In the "mathographical remarks" at the end of most chapters, you will find ample suggestions for further reading.

If things look new to you right from the beginning, check whether you know most of the concepts and symbols listed under "Basic Notations". If so, read Chapter 1, where some of the prerequisite material is reviewed. If Chapter 1 is pleasant, easy reading, then you are probably ready for this book. If more than two concepts listed under "Basic Notations" are entirely new to you, or if Chapter 1 feels challenging, then you may want to read one of the more elementary texts listed in the mathographical remarks at the end of Chapter 1.

Roughly speaking, the only prerequisite for this book is that you are at ease with set-theoretic notation. However, some knowledge of mathematical logic and (for Volume II) general topology is indispensible. Therefore, to make the exposition somewhat self-contained, we included a minicourse in mathematical logic in Chapters 5 and 6, and also an Appendix on general topology at the end of Volume II.

Once you have determined your point of entrance, it is best to read the rest of the book line by line. Much of this book is written like a dialogue between the authors and the reader. This is intended to model the practice of creative mathematical thinking, which more often than not takes on the form of an inner dialogue in a mathematician's mind. You will quickly notice that this text contains many question marks. This reflects our conviction that in the mathematical thought process it is at least as important to have a knack for asking the right questions at the right time as it is to know some of the answers.

You will benefit from this format only if you do your part and actively participate in the dialogue. This means in particular: Whenever we pose a rhetorical question, pause for a moment and ponder the question before you read our answer. Sometimes we put a little more pressure on you and call our rhetorical questions EXERCISES. Not all exercises are rhetorical questions that will be answered a few lines later. Sometimes, the completion of a proof is left as an exercise. We also may ask you to supply the entire proof of an interesting theorem, or an important example. Nevertheless, we recommend that you attempt the exercises right away, especially all the easier ones. Most of the time it will be easier to digest the ensuing text if you have worked on the exercise, even if you were unable to solve it.

Here is a well-kept secret: All mathematical research papers contain plenty of exercises. These usually appear under the disguise of seemingly unnecessary assumptions, missing examples, or phrases like: "It is easy to see." One of the most important steps in becoming a mathematician is to learn to recognize hidden EXERCISES, and to develop the habit of tackling them right away.

We often make references to solutions of exercises from earlier chapters. Sometimes, the new material will make an old and originally quite hard exercise seem trivial, and sometimes a new question can be answered by modifying the solution to a previous problem. Therefore, it is a good idea to collect your solutions and even your failed attempts at solutions in a folder where you can look them up later.

The level of difficulty of our exercises varies greatly. To help the reader save time, we rated each exercise according to what we perceive as its level of difficulty. The rating system is the same as used by American movie theatres. Everybody should attempt the exercises rated G (general audience). Beginners are encouraged to also attempt exercises rated PG (parental guidance), but may sometimes want to consult their instructor for a hint. It is also a good idea to double–check your solution with the instructor, especially if it looks trivial to you. Exercises rated R (restricted) are intended for mature audiences. The X–rated problems must not be attempted by anyone easily offended or discouraged.

There is another important reason why we do not recommend skipping chapters. Mathematical formalism is a good thing, but it is secondary to the development of the ideas that are being formalised. In the spirit of A. W. Hurwitz, we shall introduce many of the more difficult concepts in stages: first intuitively, perhaps by a suggestive analogy, and later in the text with full mathematical rigor. By skipping ahead, you may miss the more rigorous treatments of a concept and become stuck with some vague intuitive notions that you should have long outgrown. The latter problem may be alleviated by making good use of the index. Also, a footnote often alerts the reader when telling the whole truth is postponed.

If Theorem 4 of Chapter 17 is referred to in Chapter 17 itself, it will be called just Theorem 4. Outside of Chapter 17, it will be called Theorem 17.4.

The end of a proof is usually marked by a $\square$.

## Mathographical Remark

The quotes of Rota and Kac are taken from *Discrete thoughts,* by Mark Kac, Gian-Carlo Rota and Jacob T. Schwartz, Birkhäuser, Boston 1992, pages ix and 15.

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# Basic Notations

We assume that you are familiar with the concepts represented by the following symbols.
$\{x_0, x_1, \ldots, x_n\}$ — the set containing $x_0, x_1, \ldots, x_n$ and no other elements;
$\emptyset$ — the empty set;
$\in$ — the membership relation;
$\subseteq$ — subset;
$\subset$ — proper subset;
$\exists, \forall, \neg, \wedge, \vee, \rightarrow, \leftrightarrow$ — quantifiers and logical connectives. Although Chapter 5 contains a thorough discussion of these symbols in the context of formal languages, you should already be at ease with their use.
$x \cup y = \{z : z \in x \vee z \in y\}$ — the union of two sets;
$x \cap y = \{z : z \in x \wedge z \in y\}$ — the intersection of two sets;
$x \backslash y = \{z : z \in x \wedge z \notin y\}$ — the difference of two sets;
$x \Delta y = (x \backslash y) \cup (y \backslash x) = (x \cup y) \backslash (x \cap y)$ — the symmetric difference of two sets;
$\bigcup \mathcal{X} = \{z : \exists Y \in \mathcal{X}\, (z \in Y)\}$ — union of a family of sets;
$\bigcap \mathcal{X} = \{z : \forall Y \in \mathcal{X}\, (z \in Y)\}$ — intersection of a family of sets;
$f : X \to Y$ — function from $X$ into $Y$;
$f[W] = \{y \in Y : \exists x \in W\, f(x) = y\}$ — image of $W$ under $f$;
$f^{-1}Z = \{x \in X : f(x) \in Z\}$ — inverse image of $Z$ under $f$;
$dom(f)$ — domain of a function $f$;
$rng(f)$ — range of a function $f$;
$f|W$ — restriction of a function $f$ to a subset $W$ of its domain;
$f \circ g$ — composition of two functions;
$\langle a_n : n \in \mathbb{N} \rangle = (a_n)_{n \in \mathbb{N}}$ — sequence indexed by natural numbers;
$\mathbb{N} = \{0, 1, 2, \ldots\}$ — the set of natural numbers;
$\mathbb{Z}$ — the set of integers;
$\mathbb{Q}$ — the set of rationals;
$\mathbb{R}$ — the set of reals;
$\mathbb{P} = \mathbb{R} \backslash \mathbb{Q}$ — the set of irrationals;
$\mathbb{C}$ — the set of complex numbers.
$\mathbb{A}$ — the set of algebraic numbers

Note that we impose no restrictions on the style of letters that represent sets. Each of the symbols $x, X, \mathcal{X}, \mathbb{X}$ may stand for a set.

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# Subject Index

# Index of Notation

Greek letters (and $\aleph$) are listed under the first letter of their English name. Special symbols ($\emptyset$, $=$, etc.) and operations ($\alpha + \beta$, $\kappa^+$, etc.) are grouped separately at the end.

From the reviews:

Serious graduate students ... would profit from reading the book for the mathematical maturity they would gain in the process. The conversational, almost Socratic, style of exposition is well suited to giving students some insight into the process of doing mathematics as well as to the importance of asking the right questions ... Just and Weese's text would be ideally suited for... students who are serious about studying set theory.

—Journal of Symbolic Logic

The careful exposition, written in a lively and very readable style which addresses the reader rather directly, provides (by explanations, comments, and remarks) much information and motivation. Recommended.

—Monatshefte für Mathematik

This book is an introduction to set theory for beginning graduate students who want to get a sound grounding in those aspects of set theory used extensively throughout other areas of mathematics. Topics covered include formal languages and models, the power and limitation of the Axiomatic Method, the Axiom of Choice, including the fascinating Banach–Tarski Paradox, applications of Zorn's Lemma, ordinal arithmetic, including transfinite induction, and cardinal arithmetic. The style of writing, more a dialogue with the reader than that of the Master indoctrinating the pupil, makes this also very suitable for self-study.

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