Representations of Finite and Compact Groups

Barry Simon

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ABSTRACT. This book is a comprehensive pedagogical presentation of the theory of representation of finite and compact Lie groups. We discuss both the general theory and representation of specific groups. Types of groups whose representation theory is discussed include finite groups of rotations, permutation groups, and the classical compact Lie groups. Along the way, the structure theory of the compact semisimple Lie groups is exposed. The approach tends to be that of an analyst.

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To the memory of my father, Hy Simon

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INTRODUCTION

Oy! YABOGR—yet another book on group representations. The theory is so beautiful and so central to mathematics that it tends to draw authors like honey draws flies. Given the existence of several excellent monographs (among which I'd include Adams [**1**], Fulton-Harris [**5**], Samelson [**17**], and Serre [**19**]), why do I feel this book a worthy addition to the textbook literature on the subject?

I think two facets distinguish my approach. First, this book is relatively elementary; and second, while the bulk of books on the subject (and, in particular, the four quoted above) are written from the point of view of an algebraist or a geometer, this book is written with an analytical flavor.

As for the elementary nature of the material, much of it is self-contained. A prior exposure to the notions of quotient group and the isomorphism theorems is assumed; but, for example, I develop the necessary theory of algebraic integers in proving that the dimension of an irreducible representation divides the vector of the group. The material on finite groups (Chapters I–VI) should be suitable for an upper-level undergraduate course, either as a separate course or as a supplement to an advanced algebra course.

The material on compact groups is a little more sophisticated but I have discussed the needed calculus on manifolds and have even included an appendix on the basic theory of self-adjoint Hilbert-Schmidt operators, which is needed to prove the Peter-Weyl theory.

However, there are a few places we need some facts from algebraic topology that we used without proof: for example, the exact sequence of a fibration.

As for the analyst's point of view, much of the most profound work on group representations has been done by analysts: Weyl, Gel'fand, and Mackey come to mind. Indeed, this monograph bears a strong influence of George Mackey from whom I first learned much of the material thirty years ago.

The analyst's approach can be seen in several places: for example, the last three chapters discuss the structure and representations of the compact groups, *not* the representations of the semisimple Lie algebras. The two are closely related but the former is more elementary and decidedly less algebraic. In this regard, the discussion is closest to that of Adams [**1**].

A critical role is played by the fact that for compact groups, it is easy to show that any (finite-dimensional, continuous) representation supports an invariant positive definite inner product. This immediately implies that on the Lie algebra, the adjoint representation is by matrices which are skew-adjoint in a suitable inner product, so the operators are semisimple and the Killing form is (strictly) negative definite. This replaces pages of algebraic minutiae.

A good example of this philosophy is the proof in Chapter VIII that all maximal tori in a compact Lie group are conjugate and the union of all the maximal tori is the entire group. The standard proofs either go through the conjugacy of the Cartan subalgebras and considerable additional argument, or else uses Weil's approach of using the Lefschitz fixed point theorem, a sledgehammer for what is a rather simple result. Instead, I use a simple argument inductive in the dimension of G. The first 90% of the proof is that used by Varadarajan [**20**], but at a critical point, he appeals to the structure theory of the semisimple Lie algebras, which requires tens of pages of careful algebraic argument. Instead, I use the existence of an invariant inner product for the adjoint representation.

This is one of dozens of places where the proofs are ones I found while polishing the book. Nevertheless, I am not so naive as to think that there are *any* proofs here that don't appear somewhere in the literature, which is vast. But I do claim a coherent, elementary approach.

Individual chapters begin with brief summaries of what they contain. Chapter I sets the stage, focusing on counting principles as a leitmotif. The high point is the Klein-Weyl determination of the finite subgroups of three-dimensional rotations.

Chapters II–VI discuss the representations of finite groups. Chapters II–III develop the general theory and Chapters IV–VI, the representations of specific families of groups: Abelian and Clifford groups in Chapter IV, semidirect products in Chapter V, and permutation groups in Chapter VI.

The final three chapters discuss the representations of compact groups, primarily compact Lie groups. Chapter VII discusses the general theory of Lie groups and the analogs of the results of Chapter III. Chapter VIII discusses the structure theory of compact Lie groups: maximal tori, roots, and the Weyl group. It is preparation for the final chapter which presents Weyl's theory of the representations of the classical groups. The final section draws together the two halves of the book in a fascinating way by providing a proof of the Frobenius character formula for the permutation group.

By focusing on finite and compact groups, we can present the basics completely. Any attempt to go beyond this would yield a multiple-volume work (as it has in other cases!). Nevertheless, when I've given this as a one-year graduate course, I have spent five weeks discussing related topics from the representation theory of noncompact groups.

This book is based on a course I first gave at Princeton in the mid-70's. Over the ensuing twenty years, I have given the course roughly a half-dozen times at Princeton or Caltech, and each time additional polish was added. I am grateful to all the students in those courses for the feedback and insight they provided.

Parts of the actual manuscript were written during stays at the ETH-Zurich, Hebrew University, and the Technion. I appreciate their hospitality and, in particular, the courtesy shown to me by G.M. Graf, W. Hunziker, M. Ben-Artzi, and J. Avron. The preparation of the manuscript, which involved taming both TeX and my handwriting, was well handled by C. Galvez, to whom I'm grateful. I benefited from a careful reading and comments from S. Miller.

I hope you will enjoy this book. I can't think of any other course of mathematics with so much innate beauty so close to the surface.

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Barry Simon is the author of many well-known books, including such classics as *Methods of Mathematical Physics* (with M. Reed) and *Functional Integration and Quantum Physics*. This book, based on courses given at Princeton, Caltech, ETH-Zurich, and other universities, is an introductory textbook on representation theory. Two facets distinguish the approach. First, the book is relatively elementary, and second, while the bulk of the books on the subject is written from the point of view of an algebraist or a geometer, this book is written with an analytical flavor.

The exposition centers around the study of representation of certain concrete classes of groups, including permutation groups and compact semisimple Lie groups. It culminates in the complete proof of the Weyl character formula for representations of compact Lie groups and the Frobenius formula for characters of permutation groups. Extremely well tailored both for a one-year course in representation theory and for independent study, this book is an excellent introduction to the subject which is unique in having so much innate beauty so close to the surface.

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