Lectures on Elliptic and Parabolic Equations in Hölder Spaces

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ABSTRACT. The lectures concentrate on some basic facts and ideas of the modern theory of linear elliptic and parabolic equations in Hölder spaces and lead the reader as far as possible in a short course. We show that this theory, including some issues of the theory of nonlinear equations, is based on some general and extremely powerful ideas and some *simple* computations. The Sobolev-space theory, which basically follows the same lines, is left beyond the scope of the book (apart from the sharp Sobolev embedding theorem).

The main object of studies is the first boundary-value problems for elliptic and parabolic equations, with some guidelines concerning other boundary-value problems such as the Neumann or oblique derivative problems or problems involving higher-order elliptic operators acting on the boundary. Numerical approximations are also discussed.

The presentation has been chosen in such a way that after having followed the book the reader should acquire good understanding of what kinds of results are available and what kind of technique is used to obtain them.

For graduate students and scientists in mathematics, physics, and engineering interested in the theory of partial differential equations.

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Preface

In 1961, when I was a student of the fourth year at the Moscow State University (approximately the level of the second–year graduate program in the USA), I attended a regular course in the theory of partial differential equations. Later in the graduate school of the Moscow State University while dealing with some problems of optimal control of random processes, I felt a need for some results in PDEs and started to read books and articles on the subject. I was just astonished by the fact that the knowledge I got before was of no help whatsoever. Nothing like this happened with probability theory, measure theory or functional analysis. I did not even have any idea as to what constituted the framework of "modern" (at that time) theory. The basic notions of *a priori estimates* and Hölder and Sobolev spaces were not included in the standard course. The basic *method of continuity* allowing one to prove the solvability of equations on the grounds of a priori estimates was not included either. Only Laplace's equation and the heat equation were considered. What happens if we add into the equations even lower–order terms was left behind the scene. All this despite the fact that the theory of elliptic and parabolic PDEs has always been one of the basic tools in mathematics, physics, engineering and other areas of applications of mathematics.

Of course, these were the times when many important aspects of the theory were just being developed and had not yet been set down in a more or less canonical form. However, even then the well–known book by Miranda on elliptic equations and the expository article by Il'in, Kalashnikov and Oleinik on parabolic ones were already available. The excellent books by Gilbarg and Trudinger [**6**], Friedman [**7**], Ladyzhenskaya and Ural'tceva [**10**], Ladyzhenskaya, Solonnikov and Ural'tceva [**11**] came somewhat later.

Thirty–four years later in 1995 I got to teach a two–quarter course on elliptic and parabolic equations at the University of Minnesota. It is hard to describe my surprise when I found out that there was still no short albeit self–contained *textbook* for graduate students which would cover the same basic notions and answer the same basic questions without going into heavy computations. On the eve of the coming millennium I did not want to teach a course which was obsolete even 34 years ago, and this forced me to type my own lecture notes for students. The present book is a very slight extension of the lecture notes of this two–quarter course (although I added quite a few new exercises).

I have to say at once that this book is not designed as an introduction to or a guidebook on the general theory of partial differential equations. My goal was not to try to cover as many subjects as possible but rather to concentrate on some basic facts and ideas of the modern theory of elliptic and parabolic equations and to lead the reader as far as possible in a short course. The presentation has been chosen in such a way that after having followed the book, the reader should acquire a good understanding of what kinds of results are available and what kind of technique is used to obtain them. In applications this is very important knowledge, since one cannot always find in the literature all the small modifications or particular cases of known results. I also hope that after following this book the reader who decides to become a professional in the field will be well prepared for reading research articles, monographs and books such as Gilbarg and Trudinger [**6**], Friedman [**7**], Ladyzhenskaya and Ural'tseva [**10**], Ladyzhenskaya, Solonnikov and Ural'tseva [**11**], Krylov [**8**] and others containing not only mathematical results but also historical remarks and extensive bibliographies. Some of them are listed in the bibliography at the end of this book.

I set myself the task of presenting basic aspects of the theory of elliptic and parabolic equations in *Hölder spaces*, including some issues of the theory of nonlinear equations. I intend and hope to show that this theory is based on some general and extremely powerful ideas and some *simple* computations. We only deal with the Hölder–space theory of the first boundary–value problems for elliptic and parabolic equations, providing only some guidelines concerning other boundary–value problems, such as the Neumann or oblique derivative problems or problems involving higher–order elliptic operators acting on the boundary.

Very important Sobolev–space theory is left beyond the scope of this book (we only show the sharp Sobolev embedding theorem). However, the reader should not have any psychological difficulty in studying the Sobolev–space theory, since it follows the same lines and the technique is sometimes even simpler. We can recommend the very interesting recent books by Giaquinta [**4**] and Caffarelli and Cabré [**3**] bearing on this subject.

It is worth noting that usually one obtains a priori estimates in the Hölder spaces after rather lengthy, although instructive, examinations of Newtonian potentials. While treating elliptic equations in domains or parabolic equations in cylinders, one used to consider also other kinds of potentials. It turns out that application of a beautiful idea of Safonov makes the use of potentials absolutely unnecessary. Another innovation of the book is that we prove the existence theorems by using a method, introduced by Browder (1959), which does not require integral representations or Hilbert–space theories and works in the same way for domains in Euclidean spaces and for *smooth manifolds*. These two ingredients make the exposition short and self–contained and allow one to obtain the main results quickly.

Sometimes people get discouraged seeing only existence and uniqueness theorems and theorems that the solution is smoother if data is smoother. That is why we also present "real" numerical methods of approximating the solutions and show that they work owing to "abstract" existence and uniqueness results. We also discuss advantages and disadvantages of explicit formulas for solutions with regard to general theorems.

The book is designed as a textbook. Therefore it does not contain any new theoretical material but rather a new compilation of some known facts, methods and ways of presenting the material. For example, a part of Chapter 2 is close to the corresponding sections in the books by John [**5**], Gilbarg and Trudinger [**6**] or in many other books. Other parts of the book come from many other sources. The author does not make even an attempt to list all of these sources, hoping that the clear statement of not having any pretension to originality is a sufficient excuse.

My experience of following these notes at the University of Minnesota and in the Summer School at Cortona (Italy) in 1995 shows that students accept the subject very well and learn the material quickly and well. Partly this is due to exercises which have been suggested as homework.

There are about 190 exercises in the book, a few of which (about 40 marked with an *) are used in the main text. These are the simplest ones. However, many other exercises are quite difficult, despite the fact that solutions are almost always short. Therefore the reader should not feel upset if he/she cannot do them even after a good deal of thinking. Probably hints for them should have been provided right after each exercise. We do give hints to the exercises but only at the end of each chapter just to give readers an opportunity to test themselves.

Several words about notation. The index is at the end of the book. In addition, we always use the summation convention and allow constants denoted by N, usually without indices, to vary from one appearance to another even in the same proof. If we write $N = N(...)$, this means that N depends only on what is inside the parentheses. Usually in the parentheses we list the objects which are fixed. In this situation one says that the constant N is *under control*. By domains we mean general open sets. On some occasions, we allow ourselves to use different symbols for the same objects, for example,

$$u_{x^i} = \frac{\partial u}{\partial x^i} = D_i u, \quad u_x = \text{grad} u = \nabla u, \quad u_{xx} = (u_{x^i x^j}).$$

To the instructor

The book starts with Chapter 1 on elliptic equations of any order with constant (complex) coefficients in the whole space and the proof of their solvability by means of the Fourier transform. This is an encouraging start: after one or two lectures students know that a very large class of problems is solvable in a very easy way. Further material in Chapter 1 is used later in the treatment of solvability of elliptic equations of any order with variable coefficients in the whole space. Chapter 1 and the rest of the book are almost independent. The whole chapter can be dropped if one is not interested in equations of higher order or of second order with complex coefficients. Also one then has to drop Secs. 8.2 and 8.3 on "explicit formulas" for solutions of the Cauchy problem for parabolic equations. This will not affect the rest of the material related to parabolic equations.

Chapter 2 contains a traditional discussion of Laplace's equation. The contents of it should become known to every student studying the theory of PDEs because underlying ideas very often inspired further development of the theory even if only implicitly. However, if one is interested in getting to the solvability of elliptic equations with variable coefficients as soon as possible, one can actually just drop Secs. 2.1 through 2.5. Further confinement to second–order elliptic and parabolic equations with real coefficients will then allow one to start reading the book from Sec. 2.6, thus skipping about one sixth of the book. Moreover, one can then skip Secs. 3.5 through 3.7. The necessary changes in Chapter 4 are discussed in detail there. In particular, one might prefer to derive Schauder's a priori estimates for second–order elliptic equations with variable coefficients in the same way as is done in Sec. 8.9 for parabolic equations.

I had a very pleasant experience teaching a short course in the Summer School at Cortona, where in twelve lectures I covered all of Chapter 2, then Chapters 3 and 4 with the shortcuts described above, and then Chapters 5 and 6 excluding Secs. 5.5, 6.6 and 6.7. Also in that course I confined myself to $C^{2+\delta}$–theory and did not discuss the results on better regularity of solutions from Sec. 4.2.

One more way to design a short course covering elliptic *and* parabolic equations, however, is to start with Chapter 8, where we study parabolic equations, looking into the previous chapters for some proofs which are not always repeated for parabolic equations, and to extract almost all information related to elliptic equations, since for functions independent of t elliptic equations become parabolic ones (cf., for instance, Corollary 8.10.2 and Remark 8.12.3).

Finally, if one is interested only in proving solvability in Hölder spaces of *Laplace's equation*, one need not prove the interpolation inequalities in Chapter 3 and can proceed as in Sec. 8.7, where we treat the heat equation.

With such a variety of possibilities it is hard to describe prerequisites. To read the whole book, one needs to be familiar with the Fourier transform and somewhat with distributions. The necessary facts are listed in sections in which they are used. The shortest course still requires the knowledge of the theory of integration in Euclidean spaces.

Acknowledgments. I want to express my gratitude to my students at the University of Minnesota and in Cortona for their real enthusiasm, which encouraged me to compile my lecture notes into this book, and for solving many exercises which were thus checked, many of them corrected and supplied with better hints. Special thanks in this regard are due to H. Yoo.

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Nicolai Krylov
Minneapolis, October 1995

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