Fundamentals of the Theory of Operator Algebras

Volume I: Elementary Theory

Richard V. Kadison
John R. Ringrose

Graduate Studies in Mathematics
Volume 15

American Mathematical Society

http://dx.doi.org/10.1090/gsm/015

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Originally published by **Academic Press**, San Diego, California, © 1983

2000 *Mathematics Subject Classification.* Primary 46Lxx;
Secondary 46-02, 47-01.

Library of Congress Cataloging-in-Publication Data

Kadison, Richard V., 1925–
Fundamentals of the theory of operator algebras / Richard V. Kadison, John R. Ringrose.
p. cm. — (Graduate studies in mathematics, ISSN 1065-7339 ; v. 15)
Second printing, incorporating minor corrections, of the work originally published: New York : Academic Press, 1983. (Pure and applied mathematics (Academic Press) ; 100)
Includes bibliographical references and indexes.
Contents: v. 1. Elementary theory — v. 2. Advanced theory.
ISBN 0-8218-0819-2 (v. 1 : acid-free paper). — ISSN 0-8218-0820-6 (v. 2 : acid-free paper)
1. Operator algebras. I. Ringrose, John R. II. Title. III. Series.
QA326.K26 1997
512′.55—dc21 97-20916
CIP

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PREFACE

These volumes deal with a subject, introduced half a century ago, that has become increasingly important and popular in recent years. While they cover the fundamental aspects of this subject, they make no attempt to be encyclopaedic. Their primary goal is to *teach* the subject and lead the reader to the point where the vast recent research literature, both in the subject proper and in its many applications, becomes accessible.

Although we have put major emphasis on making the material presented clear and understandable, the subject is not easy; no account, however lucid, can make it so. If it is possible to browse in this subject and acquire a significant amount of information, we hope that these volumes present that opportunity—but they have been written primarily for the reader, either starting at the beginning or with enough preparation to enter at some intermediate stage, who works through the text systematically. The study of this material is best approached with equal measures of patience and persistence.

Our starting point in Chapter 1 is finite-dimensional linear algebra. We assume that the reader is familiar with the results of that subject and begin by proving the infinite-dimensional algebraic results that we need from time to time. These volumes deal almost exclusively with infinite-dimensional phenomena. Much of the intuition that the reader may have developed from contact with finite-dimensional algebra and geometry must be abandoned in this study. It will mislead as often as it guides. In its place, a new intuition about infinite-dimensional constructs must be cultivated. Results that are apparent in finite dimensions may be false, or may be difficult and important principles whose application yields great rewards, in the infinite-dimensional case.

Almost as much as the subject matter of these volumes is infinite dimensional, it is *non-commutative real analysis*. Despite this description, the reader will find a very large number of references to the "abelian" or "commutative" case—an important part of this first volume is an analysis of the abelian case. This case, parallel to function theory and measure theory, provides us with a major tool and an important guide to our intuition. A good part of what we know comes from extending to the non-commutative case results that are known in the commutative case. The "extension" process is usually difficult. The main techniques include elaborate interlacing of "abelian" segments. The reference to "real analysis" involves the fact that while we consider complex-valued functions and, non-commutatively, non-self-adjoint operators, the structures we study make simultaneously available to us the complex conjugates of those functions and, non-commutatively, the adjoints of those operators. In essence, we are studying the algebraic interrelations of systems of real functions and, non-commutatively, systems of self-adjoint operators. At its most primitive level, the non-commutativity makes itself visible in the fact that the product of a function and its conjugate is the same in either order while this is not in general true of the product of an operator and its adjoint.

In the sense that we consider an operator and its adjoint on the same footing, the subject matter we treat is referred to as the "self-adjoint theory." There is an emerging and important development of non-self-adjoint operator algebras that serves as a non-commutative analogue of complex function theory—algebras of holomorphic functions. This area is not treated in these volumes. Many important developments in the self-adjoint theory—both past and current—are not treated. The type I C^*-algebras and C^*-algebra K-theory are examples of important subjects not dealt with. The aim of teaching the basics and preparing the reader for individual work in research areas seems best served by a close adherence to the "classical" fundamentals of the subject. For this same reason, we have not included material on the important application of the subject to the mathematical foundation of theoretical quantum physics. With one exception, applications to the theory of representations of topological groups are omitted. Accounts of these vast research areas, within the scope of this treatise, would be necessarily superficial. We have preferred instead to devote space to clear and leisurely expositions of the fundamentals. For several important topics, two approaches are included.

Our emphasis on instruction rather than comprehensive coverage has led us to settle on a very brief bibliography. We cite just three textbooks (listed as [H], [K], and [R]) for background information on general topology and measure theory, and for this first volume, include only 25 items from the literature of our subject. Several extensive and excellent bibliographies are available (see, for example, [2,24,25]), and there would be little purpose in reproducing a modified version of one of the existing lists. We have included in our references items specifically referred to in the text and others that might provide profitable additional reading. As a consequence, we have made no attempt, either in the text or in the exercises, to credit sources on which we have drawn or to trace the historical background of the ideas and results that have gone into the development of the subject.

Each of the chapters of this first volume has a final section devoted to a substantial list of exercises, arranged roughly in the order of the appearance of topics in the chapter. They were designed to serve two purposes: to illustrate and extend the results and examples of the earlier sections of the chapter, and to help the reader to develop working technique and facility with the subject matter of the chapter. For the reader interested in acquiring an ability to work with the subject, a certain amount of exercise solving is indispensable. We *do not* recommend a rigid adherence to order—each exercise being solved in sequence and no new material attempted until all the exercises of the preceding chapter are solved. Somewhere between that approach and total disregard of the exercises a line must be drawn congenial to the individual reader's needs and circumstances. In general, we *do* recommend that the greater proportion of the reader's time be spent on a thorough understanding of the main text than on the exercises. In any event, *all* the exercises have been designed to be solved. Most exercises are separated into several parts with each of the parts manageable and some of them provided with hints. Some are routine, requiring nothing more than a clear understanding of a definition or result for their solutions. Other exercises (and groups of exercises) constitute small (guided) research projects.

On a first reading, as an introduction to the subject, certain sections may well be left unread and consulted on a few occasions as needed. Section 2.6, *Tensor products and the Hilbert–Schmidt class* (this "subsection" is the largest part of Section 2.6) will not be needed seriously until Chapter 11 (in Volume II). All the material on unbounded operators (and the material related to Stone's theorem) will not be needed until Chapter 9 (in Volume II). Thus Section 2.7, Section 3.2, *The Banach algebra* $L_1(\mathbb{R})$ *and Fourier analysis*, the last few pages of Chapter 4 (including Theorem 4.5.9), and Section 5.6, can be deferred to a later reading. Some readers, more or less familiar with the elements of functional analysis, may want to enter the text after Chapter 1 with occasional back references for notation or precise definitions and statements of results. The reader with a good general knowledge of basic functional analysis may consider beginning at Section 3.4 or perhaps with Chapter 4.

The various possible styles of reading this volume, related to the levels of preparation of the reader, suggest several styles and levels of courses for which it can be used. For all of these, a good working knowledge of point-set (general) topology, such as may be found in [K], is assumed. Somewhat less vital, but useful, is a knowledge of general measure theory, such as may be found in [H] and parts of [R]. Of course, full command of the fundamentals of real and complex analysis (we refer to [R] for these) is needed; and, as noted earlier, the elements of finite-dimensional linear algebra are used. The first three chapters form the basis of a course in elementary functional analysis with a slant toward operator algebras and its allied fields of group representations, harmonic analysis, and mathematical (quantum) physics. These chapters provide material for a brisk one-semester course at the first- or second-year graduate level or for a more leisurely one-year course at the advanced undergraduate or beginning graduate level. Chapters 3, 4, and 5 provide an introduction to the theory of operator algebras and have material that would serve as a one-semester graduate course at the second- or third-year level (especially if Section 5.6 is omitted). In any event, the book has been designed for individual study as well as for courses, so that the problem of a wide spread of preparation in a class can be dealt with by encouraging the better prepared students to proceed at their own paces. Seminar and reading-course possibilities are also available.

When several (good) terms for a mathematical construct are in common use, we have made no effort to choose one and then to use that one term consistently. On the contrary, we have used such terms interchangeably after introducing them simultaneously. This seems the best preparation for further reading in the research literature. Some examples of such terms are weaker, coarser (for topologies on a space), unitary transformation, and Hilbert space isomorphism (for structure-preserving mappings between Hilbert spaces). In cases where there is conflicting use of a term in the research literature (for example, "purely infinite" in connection with von Neumann algebras), we have avoided all use of the term and employed accepted terminology for each of the constructs involved. Since the symbol * is used to denote the adjoint operations on operators and on sets of operators, we have preferred to use a different symbol in the context of Banach dual spaces. We denote the dual space of a Banach space $\mathfrak{X}$ by $\mathfrak{X}^\sharp$. However, we felt compelled by usage to retain the terminology "weak *" for the topology induced by elements of $\mathfrak{X}$ (as linear functionals on $\mathfrak{X}^\sharp$).

Results in the body of the text are italicized, titled Theorem, Proposition, Lemma, and Corollary (in decreasing order of "importance"—though, as usual, the "heart of the matter" may be dealt with in a lemma and its most usable aspect may appear in a corollary). In addition, there are Remarks and Examples that extend and illuminate the material of a section, and of course there are the (formal) Definitions. None of these items is italicized, though a crucial phrase or word frequently is. Each of these segments of the text is preceded by a number, the first digit of which indicates the chapter, the second the section, and the last one- or two-digit number the position of the item in the section. Thus, "Proposition 5.5.18" refers to the eighteenth numbered item in the fifth section of the fifth chapter. A back or forward reference to such an item will include the title ("Theorem," "Remark," etc.), though the number alone would serve to locate it. Occasionally a displayed equation, formula, inequality, etc., is assigned a number in parentheses at the left of the display—for example, the "convolution formula" of Fourier transform theory appears as the display numbered (4) in the proof of Theorem 3.2.26. In its own section, it is referred to as (4) and elsewhere as 3.2(4).

The lack of illustrative examples in much of Chapter 1 results from our wish to bring the reader more rapidly to the subject of operator algebras rather than to dwell on the basics of general functional analysis. As compensation for their lack, the exercises supply much of the illustrative material for this chapter. Although the tensor product development in Section 2.6 may appear somewhat formal and forbidding at first, it turns out that the trouble and care taken at that point simplify subsequent application. The same can be said (perhaps more strongly) about Section 5.6. The material on unbounded operators (their spectral theory and function calculus) is so vital when needed and so susceptible to incorrect and incomplete application that it seemed well worth a careful and thorough treatment. We have chosen a powerful approach that permits such a treatment, much in the spirit of the theory of operator algebras.

Another (general) aspect of the organization of material in a text is the way the material of the text proper relates to the exercises. As a matter of specific policy, we have not relegated to the exercises whole arguments or parts of arguments. Reference is occasionally made to an exercise as an illustration of some point—for example, the fact that the statement resulting from the omission of some hypothesis from a theorem is false.

During the course of the preparation of these volumes, we have enjoyed, jointly and separately, the hospitality and facilities of several universities, aside from our home institutions. Notable among these are the Mathematics Institutes of the Universities of Aarhus and Copenhagen and the Theoretical Physics Institute of Marseille–Luminy. The subject matter of these volumes and its style of development is inextricably interwoven with the individual research of the authors. As a consequence, the support of that research by the National Science Foundation (U.S.A.) and the Science Research Council (U.K.) has had an oblique but vital influence on the formation of these volumes. It is the authors' pleasure to express their gratitude for this support and for the hospitality of the host institutions noted.

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Praise for both volumes …

… these two volumes represent a magnificent achievement. They will be an essential item on every operator algebraist's bookshelves and will surely become the primary source of instruction for research students in von Neumann algebra theory.

—*Bulletin of the London Mathematical Society*

Volumes I and II were published in 1982 and 1983. Since then they have quickly established themselves as The Textbooks in operator algebra theory.

—*Bulletin of the American Mathematical Society*

One of the splendid features of the original two volumes is their large supply of exercises … which illustrate the results of the text and expand its scope.

—*L'Enseignement mathématique*

This work, together with *Fundamentals of the Theory of Operator Algebras. Volume II: Advanced Theory*, Graduate Studies in Mathematics, vol. 16, presents an introduction to functional analysis and the initial fundamentals of C^*- and von Neumann algebra theory in a form suitable for both intermediate graduate courses and self-study. The authors provide a clear account of the introductory portions of this important and technically difficult subject. Major concepts are sometimes presented from several points of view; the account is leisurely when brevity would compromise clarity. An unusual feature in a text at this level is the extent to which it is self-contained; for example, it introduces all the elementary functional analysis needed. As the emphasis is on teaching the subject, the book is well-supplied with exercises. With the only prerequisites necessary to use the book being basic measure theory and topology, the book presents the possibility for the design of numerous courses aimed at different audiences.

ISBN 978-0-8218-0819-1

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