Fundamentals of the Theory of Operator Algebras

Volume II: Advanced Theory

Richard V. Kadison
John R. Ringrose

Graduate Studies in Mathematics
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PREFACE

Most of the comments in the preface appearing at the beginning of Volume I are fully applicable to this second volume. This is particularly so for the statement of our primary goal: to *teach* the subject rather than be encyclopaedic. Some of those comments refer to possible styles of reading and using Volume I. The reader who has studied the first volume following the plan that avoids all the material on unbounded operators can continue in this volume, deferring Lemma 6.1.10, Theorem 6.1.11, and Theorem 7.2.1′ with its associated discussion to a later reading. This program will take the reader to Section 9.2, where Tomita's modular theory is developed. At that point, an important individual decision should be made: Is it time to retrieve the unbounded operator theory or shall the first reading proceed without it? The reader can continue without that material through all sections of Chapters 9 (other than Section 9.2), 10, 11, and 12 (ignoring Subsection 11.2, *Tensor products of unbounded operators*, which provides an alternative approach to the commutant formula for tensor products of von Neumann algebras). However, avoiding Section 9.2 makes a large segment of the post-1970 literature of von Neumann algebras unavailable. Depending on the purposes of the study of these volumes, that might not be a workable restriction. Very little of Chapter 13 is accessible without the results of Section 9.2, but Chapter 14 can be read completely.

Another shortened path through this volume can be arranged by omitting some of the alternative approaches to results obtained in one way. For example, the first subsection of Section 9.2 may be read and the last two omitted on the first reading. The last subsection of Section 11.2 may also be omitted. It is not recommended that Section 7.3 be omitted on the first reading although it does deal primarily with an alternative approach to the theory of normal states. Too many of the results and techniques appearing in that section reappear in the later chapters. Of course, all omissions affect the exercises and groups of exercises that can be undertaken.

As noted in the preface appearing in Volume I, certain exercises (and groups of exercises) "constitute small (guided) research projects." Samples of this are: the Banach–Orliz theorem developed in Exercises 1.9.26 and 1.9.34; the theory of compact operators developed in Exercises 2.8.20–2.8.29, 3.5.17, and 3.5.18; the theory of $\beta(\mathbb{N})$ developed in Exercises 3.5.5, 3.5.6, and 5.7.14–5.7.21. There are many other such instances. To a much greater extent, this process was used in the design of exercises for the present volume; results on diagonalizing abelian, self-adjoint families of matrices over a von Neumann algebra are developed in Exercises 6.9.14–6.9.35; the algebra of unbounded operators affiliated with a finite von Neumann algebra is constructed in Exercises 6.9.53–6.9.55, 8.7.32–8.7.35, and 8.7.60. The representation-independent characterizations of von Neumann algebras appear in Exercises 7.6.35–7.6.45 and 10.5.85–10.5.87. The Friedrichs extension of a positive symmetric operator affiliated with a von Neumann algebra is described in Exercises 7.6.52–7.6.55, and this topic is needed in the development of the theory of the positive dual and self-dual cones associated with von Neumann algebras that appears in Exercises 9.5.51–9.6.65. A detailed analysis of the intersection with the center of various closures of the convex hull of the unitary conjugates of an operator in a von Neumann algebra is found in Exercises 8.7.4–8.7.22, and the relation of these results to the theory of conditional expectations in von Neumann algebras is the substance of the next seven exercises; this analysis is also applied to the development of the theory of (bounded) derivations of von Neumann algebras occurring in Exercises 8.7.51–8.7.55 and 10.5.76–10.5.79. Portions of the theory of representations of the canonical anticommutation relations appear in Exercises 10.5.88–10.5.90, 12.5.39, and 12.5.40. This list could continue much further; there are more than 1100 exercise tasks apportioned among 450 exercises in this volume. The index provides a usable map of the topical relation of exercises through key-word references.

Each exercise has been designed, by arrangement in parts and with suitable hints, to be realistically capable of solution by the techniques and skills that will have been acquired in a careful study of the chapters preceding the exercise. However, full solutions to all the exercises in a topic grouping may require serious devotion and time. Such groupings provide material for special seminars, either in association with a standard course or by themselves. Seminars of that type are an invaluable "hands-on" experience for active students of the subject.

Aside from the potential for working seminars that the exercises supply, a fast-paced, one-semester course could cover Chapters 6–9. The second semester might cover the remaining chapters of this volume. A more leisurely pace might spread Chapters 6–10 over a one-year course, with an expansive treatment of modular theory (Section 9.2) and a careful review (study) of the unbounded operator theory developed in Sections 2.7 and 5.6 of Volume I. Chapters 11–14 could be dealt with in seminars or in an additional semester course. In addition to these course possibilities, both volumes have been written with the possibility of self-study very much in mind.

The list of references and the index in this volume contain those of Volume I. Again, the reference list is relatively short, for the reasons mentioned in the preface in Volume I. A special comment must be made about the lack of references in the exercise sections. Many of the exercises (especially the topic groupings) are drawn from the literature of the subject. In designing the exercises (parts, hints, and formulation), complete, model solutions have been constructed. These solutions streamline, simplify, and unify the literature on the topic in almost all cases; on occasion, new results are included. References to the literature in the exercise sets could misdirect more than inform the reader. It seems expedient to defer references for the exercises to volumes containing the exercises and model solutions; a significant number of references pertain directly to the solutions. We hope that the benefits from the more sensible references in later volumes will outweigh the present lack; our own publications have been one source of topic groupings subject to this policy.

Again, individual purposes should play a dominant role in the proportion of effort the reader places on the text proper and on the exercises. In any case, a good working procedure might be to include a careful scanning of the exercise sets with a reading of the text even if the decision has been made not to devote significant time to solving exercises.

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CONTENTS OF VOLUME I

PREFACE TO THE SECOND PRINTING

Minor corrections, noted during the ten years since the publication of this volume, are the only changes made to the original volume. A list of these corrections are appended to this preface for the convenience of readers with a copy of the original printing.

One mathematical point brought to our attention by Paul Halmos deserves special mention. It involves the polar decomposition of a normal operator. With T normal, $(T^*T)^{1/2} = (TT^*)^{1/2}(= H)$. Thus, as noted on page 402 (following the proof of Proposition 6.1.3), $UH = T = HU$, where UH and HU are the ("left" and "right") polar decompositions of T. (Use is made of Theorem 6.1.2 here.) We follow that observation with the assertion, "Conversely, from the uniqueness of the polar decomposition (left and right), if $UH = HU$, $(T^*T)^{1/2} = (TT^*)^{1/2}$ and $T^*T = TT^*$." Our intention here is to make, in economical form, the assertion: If T has left polar decomposition UH and right polar decomposition HU (so that, of course, $UH = HU$), then T is normal. This is correct, and the few words of argument given demonstrate that. Unfortunately, the economy of our statement leads to difficulty, for the reader may quite naturally interpret the stated assertion as: If T has (left) polar decomposition UH, and $UH = HU$, then T is normal. This is false, in general — for example, if T is an isometry of a Hilbert space onto a proper subspace.

Although no new topics have been added in this second printing, several could well have been included. In the intervening ten years, some topics have proved themselves to be of fundamental significance to the study of operator algebras. First and foremost among these is Alain Connes's brilliant development of the subject that has come to be known as "Noncommutative Geometry" (compare his "Noncommutative Geometry." Academic Press, San Diego, 1994).

The results that each abelian C*-algebra $\mathfrak{A}$ is * isomorphic with $C(X)$, where X is the compact Hausdorff space of pure states of $\mathfrak{A}$ (topologized by the induced weak* topology of the Banach dual space of $\mathfrak{A}$) and that two compact Hausdorff spaces X and Y are homeomorphic if and only if $C(X)$ and $C(Y)$ are algebraically isomorphic by an isomorphism implemented by the homeomorphism (Theorems 4.4.3 and 3.4.3) make it clear that the topology of a compact Hausdorff space is encoded in the algebraic structure of its function algebra $C(X)$. To illustrate this in a simple way, X fails to be connected precisely when $C(X)$ contains an idempotent function f (so, $f = f^2$) distinct from 0 and 1. At the same time, each $C(X)$ is an abelian C*-algebra.

The identification of abelian C*-algebras with function algebras ($C(X)$) leads, at once, to the interpretation of a non-commutative C*-algebra $\mathfrak{A}$ as the (non-commuting) "function algebra" of some "non-commutative" (compact Hausdorff) space. From the comments made before, we can readily imagine that the "management" of the projections in $\mathfrak{A}$ is intimately related to the "connectivity properties" of the underlying non-commutative space. That system of projection management is a primary aspect of the subject that has become known as C*-algebra K-theory.

The discussion, to this point, involves the topology of the non-commutative space underlying $\mathfrak{A}$. Other geometric aspects of that space require notions of integration and differentiation in the "function algebra" $\mathfrak{A}$. The integration can be supplied by appropriate states of $\mathfrak{A}$; the differentiation makes use of "derivations" of (dense subalgebras of) $\mathfrak{A}$. The theory of (bounded) derivations of operator algebras is introduced in Exercise 4.6.65 and developed further throughout the exercises of this volume (notably, in the exercises of Chapters 8 and 10). The theory of unbounded (densely defined) derivations, developed largely by Shôichiro Sakai, is described in his profound "Operator Algebras in Dynamical Systems." (Cambridge University Press, Cambridge 1993).

In Chapter 5, we introduced von Neumann algebras and factors (preceding Example 5.1.6) and proved some basic facts about them. Much of this second volume develops the more advanced theory of von Neumann algebras. At one point, we define the fascinating class of "factors of type II_1" (Definition 6.5.1 and Corollary 6.5.3). As it turns out, they are precisely the infinite-dimensional factors that admit a "tracial state" (a state τ with the property that $\tau(AB) = \tau(BA)$ for all A and B in the factor — compare Section 8.2). The factors of type II_1 share some of the properties of the other class of "finite" factors, the algebras $\mathcal{B}(\mathcal{H})$ with $\mathcal{H}$ a finite-dimensional Hilbert space. (For example, in each finite factor, if an operator has a left inverse, it has an inverse.) Many of the phenomena that occur in discrete steps for the finite factors of "type I" (isomorphic to some $\mathcal{B}(\mathcal{H})$) appear in the case of factors of type II_1 and occur "continuously." An instance of this is seen with the "dimension function" that assigns to each projection in $\mathcal{B}(\mathcal{H})$ the dimension of its range. Rescaled so that I has "dimension" 1, this dimension function assigns the values 0, $1/m$, $2/m$, ..., $(m-1)/m$, 1 to the projections of $\mathcal{B}(\mathcal{H})$, when $\mathcal{H}$ is m-dimensional. This dimension function, characterized by a few simple properties, is unique. Characterized by these same properties, there is a unique dimension function defined on the projections in a factor of type II_1 (compare Section 8.4). This dimension function assigns all the values in $[0,1]$ to the projections of the factor. The phenomenon of (finite) "continuous dimensionality" is the essence of factors of type II_1.

If the Hilbert space $\mathcal{H}$ has dimension nm, with n and m positive integers, and $\mathcal{M}$ is a subfactor of $\mathcal{B}(\mathcal{H})$ isomorphic to $\mathcal{B}(\mathcal{H}_1)$, where $\mathcal{H}_1$ has dimension n, then the commutant $\mathcal{M}'$ of $\mathcal{M}$ is isomorphic to $\mathcal{B}(\mathcal{H}_2)$, where $\mathcal{H}_2$ has dimension m. If Δ and Δ' are the dimension functions on $\mathcal{M}$ and $\mathcal{M}'$, respectively, and E and E' are minimal projections in $\mathcal{M}$ and $\mathcal{M}'$, respectively, then $\Delta(E) = 1/n = (m/n)\Delta'(E')$, If $\mathcal{K}$ is infinite dimensional and $\mathcal{N}$ is a subfactor of $\mathcal{B}(\mathcal{K})$ of type II_1, $\mathcal{N}'$ is its commutant, F is a projection in $\mathcal{N}$ with range $[\mathcal{N}'u]$ for some unit vector u in $\mathcal{K}$, and F' is the projection in $\mathcal{N}'$ with range $[\mathcal{N}u]$, then $\mathcal{N}'$ is a factor either of type II_1, again, or of type II_∞ (compare Definition 6.5.1). In case $\mathcal{N}'$ is of type II_1, there is a constant c, independent of the vector u, such that $\Delta(F) = c\Delta'(F')$, where Δ and Δ' are the dimension functions on $\mathcal{N}$ and $\mathcal{N}'$, respectively (compare Exercise 9.6.5). This "coupling constant" was introduced by Murray and von Neumann. For the case of the finite factors of type I, where $\mathcal{H}$ has dimension nm, the corresponding coupling constant is m/n; any positive rational number can occur. With $\mathcal{N}$ and $\mathcal{N}'$ factors of type II_1, the coupling constant may be any positive real number.

It is always possible to represent a factor $\mathcal{M}$ of type II_1 (isomorphically) on a Hilbert space $\mathcal{H}$ so that $\mathcal{M}'$ is of type II_1 and the coupling constant is 1. In this case, there is a unit vector u generating for both $\mathcal{M}$ and $\mathcal{M}'$. The GNS construction applied to the tracial state of $\mathcal{M}$ provides such a representation of $\mathcal{M}$ (compare Lemma 7.2.14). If $\mathcal{M}$ is so represented and $\mathcal{N}$ is a subfactor of $\mathcal{M}$ of type II_1, then $\mathcal{M}' \subseteq \mathcal{N}'$ and $c \geq 1$, where c is the coupling constant from $\mathcal{N}$ to $\mathcal{N}'$. A puzzling question that goes back to 1950 (though not written about) asks which values of c can occur.

With an impressive display of ingenuity, technique, perseverance, and entrepreneurial skill, Vaughan Jones resolves this puzzle in "Index for subfactors," Invent. Math. 72(1983), 1–25. Jones proves the remarkable result that c, which he calls the "index" of $\mathcal{N}$ in $\mathcal{M}$ can (and does) take all the values $4cos^2\frac{\pi}{n}$ in $[1,4)$ as n assumes the integer values 3,4,..., as well as all the values in $[4,\infty)$. The key to his proof is the projection E_1 from $\mathcal{H}$ onto $[\mathcal{N}u]$, which is the geometric representation of the "conditional expectation mapping" of $\mathcal{M}$ onto $\mathcal{N}$ (compare Exercises 8.7.23 and 10.5.86). The von Neumann algebra $\mathcal{M}_1$ generated by $\mathcal{M}$ and E_1 is, again, a factor of type II_1 and "the Jones index" of $\mathcal{M}$ in $\mathcal{M}_1$ is the same as that of $\mathcal{N}$ in $\mathcal{M}$. Repeating this procedure, there is a projection E_2 (the geometric representation of the conditional expectation of $\mathcal{M}_1$ onto $\mathcal{M}$). Continuing in this way, we construct the sequence of "Jones projections," $E_1, E_2, \ldots$, and the "Jones tower" of factors $\mathcal{N}, \mathcal{M}, \mathcal{M}_1, \mathcal{M}_2, \ldots$ of type II_1. The projections $E_1, E_2, \ldots$ satisfy certain relations, which imply that each finite subset generates a finite-dimensional algebra. The relations involve the index of $\mathcal{N}$ in $\mathcal{M}$, intimately, and ultimately provide the surprising restrictions on the possible values of the index noted in the theorem of Jones. At the same time, Jones notes a close connection between his relations and "braid relations." Using this connection, the tower of factors of type II_1, and the (unique) tracial states on these factors, Jones constructs a polynomial invariant for knots and applies it to the solution of old problems in knot theory. Aspects of the Jones index theory have found their way to statistical physics and fundamental biology. Within the subject of this volume, the index of $\mathcal{N}$ in $\mathcal{M}$ is an invariant of $\mathcal{N}$ under automorphisms of $\mathcal{M}$ and has led to the area of classification of subfactors of a factor of type II_1 (up to automorphisms of the factor). As this is written, there are already an array of deep results in this area. Certainly, the developments around the Jones index form one of the glorious chapters in operator-algebra research.

Throughout the preceding volume, we have developed the example of a normal operator as a "multiplication operator" acting on the L_2-space of a measure space (compare Examples 2.4.11 and 2.5.12) and the corresponding example of the "algebra of bounded multiplication operators." In Example 5.1.6, we note that this algebra is an abelian von Neumann algebra and, indeed, "maximal abelian." In Theorem 9.4.1, we establish a converse to this. Up to isomorphism, then, abelian von Neumann algebras are measure algebras. In the same way that the study of abelian C*-algebras is the general framework for (classical) analysis, the study of abelian von Neumann algebras is the general framework for classical measure theory. As the study of non-commutative C*-algebras is "non-commutative analysis," the study of non-commutative von Neumann algebras is "non-commutative measure theory." This point of view makes itself apparent throughout the subject of von Neumann algebras.

Since probability and statistics are so closely related to the language and results of measure theory, the results and concepts of probability and statistics often have non-commutative analogues stated in the language of (non-commutative) von Neumann algebras. "Independence" in these subjects has commutativity as its basic requirement. As long as type I factors (von Neumann algebras) form the background for the discussion, commutativity will suffice as the main component of statistical independence. Analytic subtleties require something more restrictive when factors of other types are involved. The stronger "independence" that comes from "tensorial splitting" is usually what is needed. (Tensor products of operator algebras are studied in Chapter 11.)

Dan Voiculescu creates a truly non-commutative notion of independence by replacing commutativity (tensorial and otherwise) by "freeness." Loosely speaking, for "free" independence, A and B must be generators of a "free" (non-commutative) algebra. Of course, there are analytic requirements in this setting. He then develops the probability and statistics corresponding to "free independence." In this case, a "semi-circular distribution" takes the place of the usual Gaussian distribution. Examples of factors in which freeness plays a prominent role are introduced in Section 6.7. Countable (discrete) groups in which the conjugacy class of each element, other than the unit, is infinite (*i.c.c groups*) give rise to factors of type II_1 (one of the natural "group algebras" of the group — compare Theorem 6.7.5). The free groups $\mathcal{F}_n$ on n (> 1) generators provide specific examples of i.c.c groups. Since 1950, the problem of whether or not the associated factors $\mathcal{L}_{\mathcal{F}_n}$ of type II_1 are isomorphic has been one of the most vexing "yes-or-no" questions. It is open, as this is written. At the same time, the algebra of finite matrices of order m with entries from a given factor of type II_1 is, again, a factor of type II_1. Is it isomorphic to the original factor? Murray and von Neumann raised this question in [58]. It, too, is open at this writing, though deep work of Alain Connes sheds considerable light on it (and settles an allied problem). In a breathtaking *tour de force*, Voiculescu uses his "free probability" theory to make significant inroads into the isomorphism problem for the free-group factors. He proves, for example, that $\mathcal{L}_{\mathcal{F}_2}$ is isomorphic to the 2×2 matrix algebra with entries from $\mathcal{L}_{\mathcal{F}_5}$.

In a related development, Voiculescu defines a concept of "free entropy," in his free probability framework, and uses it to show that the factors $\mathcal{L}_{\mathcal{F}_n}$ do not contain certain types of maximal abelian subalgebras (answering a longstanding question). Using Voiculescu's free entropy, Liming Ge answers an even older question about the maximal abelian subalgebras of these factors. Again, using free entropy, Ge settles a difficult and fascinating problem by showing that the factors $\mathcal{L}_{\mathcal{F}_n}$ are not the tensor product of two factors of type II_1.

ERRATA TO THE FIRST PRINTING

p. 402 line 14 A footnote is inserted here '...Conversely†, from uniqueness...' The footnote should read: † See the second paragraph of the preface to the second printing.
p. 431 line −9 'E_{b_0,b_0}' for '$E_{b_0,b}$'
p. 456 line −16 'η' for 'n'
p. 523 line −5 '*contains a non-zero*' for 'contains a non-zero' (italics)
p. 713 line −15 '$\mathcal{B}(\mathcal{H}_\Phi)$' for '$\mathcal{B}(\mathcal{H})$'
p. 768 line −12 'approximate' for 'approximately' and 'identity' for 'identiy'
p. 773 line 12 'mappings' for 'mapppings'
p. 779 line −11 '.]' for '].'
p. 784 line −2 '$(A \in \mathfrak{A})$' for '$(A \in \mathcal{R})$'
p. 785 line 11 'Use Exercises' for 'Use Exercise'
p. 795 line 7 '$B_1 \cdots B_j$' for '$B_1, \dots, B_j$'
p. 795 line 8 '$B_1 \cdots B_n$' for '$B_1, \dots, B_n$'
p. 806 line 4 '$A_{1j} \otimes \cdots \otimes A_{nj}$' for '$A_1 \otimes \cdots \otimes A_n$'
p. 818 lines 13, 14 read 'extension' and 'C*-algebra' (bad letters)
p. 857 line −14 '$\mathcal{B}$' for 'β'
p. 878 line 16 '11.5.3(iii)' for '11.5.3(ii)'
p. 900 lines 18, 19, 20, 21 missing letters at line ends: 'and' 'we' 'so' 'and'
p. 916 line −5 poor absolute value bar
p. 924 line 13 insert 'such that $\omega(E) > 0$' after 'normal state of $\mathcal{R}$'
p. 931 line −19 '$(\rho \bar{\otimes} \sigma)$' for '$(\rho \otimes \sigma)$'
p. 931 line −18 '$(\rho \bar{\otimes} \sigma)$' for '$(\rho \otimes \sigma)$' (twice)
p. 931 line −14 'tively' broken 't'
p. 931 line −13 '$(\rho' \bar{\otimes} \sigma)$' for '$(\rho' \otimes \sigma)$'
p. 931 line −13 '$(\rho \bar{\otimes} \sigma')$' for '$(\rho \otimes \sigma')$'
p. 977 line 5 '$I \otimes l_t$' for '$I \bar{\otimes} l_t$'
p. 990 line 18 '$\mathcal{B}(\mathcal{K})$' for '$\mathcal{B}(K)$'
p. 990 line 20 'is a *' for 'is *'
p. 993 line 5 '$\sum_{n=3}^{\infty}$' for '$\sum_{n=1}^{\infty}$'
p. 1046 line 9 'Lebesgue' for 'Lebsegue'

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