Topics in Classical Automorphic Forms

Henryk Iwaniec

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Preface

Automorphic forms are present in almost every area of modern number theory. They also appear in other areas of mathematics and in physics. I have lectured on these topics many times at Rutgers University with only slight overlapping of content, and I still have new material to teach that is important. It is indeed a vast territory which cannot be grasped by any one person. While research publications on automorphic forms are rapidly increasing in quantity and quality, the demand for textbooks, particularly on the graduate level, is also growing. There are fine books on this subject such as [Lan], [Miy], [Sh2], but still more are needed, especially those which favor analytic methods.

The present book is based on my lecture notes (almost verbatim except for Section 5.5) from a graduate course which I delivered in the Fall of 1994 and in the Spring of 1995 at Rutgers. The course was formulated, as the title implies, to acquaint our new students with the subject matter from various perspectives. Thus I have not followed direct or traditional paths, but rather I have frequently ventured into areas where different ideas and methods mix and interact. To cover a lot in a limited time, some material is necessarily presented as a survey. For example, the numerous connections of automorphic forms with $L$-functions of number fields are discussed in Chapter 12 without details. However we do provide complete proofs of the most basic results in the earlier sections.

An experienced reader will find some of our arguments to be nonstandard. It would be pointless to argue which approach is better, since our choice was made simply for the purpose of showing different possibilities. For example, our presentation of the theory of Hecke operators in Chapter 6
is completed for primitive characters quickly by establishing the multiplicity-one principle using Gauss and Ramanujan sums instead of lengthy considerations of inner products. Of course, this is only a special case (the whole space is spanned by newforms), but it is an important case.

We pay great attention to detail in the subjects of theta functions and representations by quadratic forms (Chapters 10 and 11) because these are not sufficiently covered in textbooks, despite having a long history of research.

Because the original notes where written as the course was progressing, inevitably some redundancy has occurred. Nevertheless we have decided not to eliminate this redundancy, because it offers the option of selective reading. For example, our account of the Shimura-Taniyama conjecture for special curves (the congruent number curves) is self-contained in Chapter 8, even though one could instead appeal to the later chapters on general theta functions.

Sergei Gelfand, Peter Sarnak and others have convinced me that these lecture notes might be useful for a large number of graduate students, and they have urged me to publish them. I would like to thank them for their encouragement. I am grateful to W. A. Gonzalez, C. L. Hamer and C. J. Moz-zochi for helping me in the technical preparation of the original notes. Special thanks are expressed to T. Khovanova for corrections and improvements which she contributed when editing these notes for publication.

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The main purpose of the book is to present the reader with various perspectives of the theory of automorphic forms. In addition to detailed and often nonstandard exposition of familiar topics of the theory, with a particular emphasis on analytic aspects, special attention is paid to such subjects as theta-functions and representations of integers by quadratic forms.