

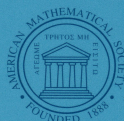
Growth of Algebras and Gelfand-Kirillov Dimension

REVISED EDITION

Günter R. Krause
Thomas H. Lenagan

**Graduate Studies
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Providence, Rhode Island

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ABSTRACT. The Gelfand-Kirillov dimension measures the asymptotic rate of growth of algebras. Since it provides important structural information, this invariant has become one of the standard tools in the study of infinite dimensional algebras. This book gives a systematic treatment of the basic properties of Gelfand-Kirillov dimension and presents applications to various areas, such as Weyl algebras, universal enveloping algebras of finite dimensional Lie algebras, polynomial identity algebras, and groups.

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Preface

During the two decades that preceded the publication of the first edition of this book, [101], the Gelfand-Kirillov dimension had emerged as a very useful and powerful tool for investigating noncommutative algebras. Since the basic ideas and results required to work with this concept were scattered over various journal articles, the need arose for providing a coherent and reliable source of information for researchers working in this area. This gave the motivation for writing the earlier version of this book. Since it has become a standard reference, we have incorporated the original text into the second edition with only minor modifications. In particular, the numbering of theorems, lemmas, etc., has not been changed. Errors that we have become aware of have been corrected, quite a few items have been rephrased, and more mathematical expressions have been displayed for better clarity. Otherwise, the reader familiar with the first edition will find that it is virtually identical with the first eleven chapters of the second one.

Since 1984, many articles have been published on this subject, and a detailed account of even a small portion of all the work that has been done would have greatly exceeded the scope of this text. Thus, in the added Chapter 12, we provide for the most part only sketches of the new developments that have surfaced in the last few years, referring to the literature for details. The bibliography has been updated accordingly, it is now almost twice the size of the original one.

We wish to express our gratitude to the many mathematicians with whom we were able to discuss the ideas presented in this book. Special thanks are due to Paul Smith for numerous suggestions and detailed criticism, and to John McConnell for his advice and for allowing us to use

material from various manuscripts that had not yet been published at the time the first edition went into print.

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On the technical side, we are greatly indebted to Helena Cameron of the University of Edinburgh who typed the manuscript for the first edition, and who, more than ten years later, prepared the base L^AT_EX 2_ε version of the current Chapters 1–11. We are also grateful to Michael Doob and Craig Platt of the University of Manitoba for providing assistance with the electronic typesetting of this book.

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