A Course in Differential Geometry

Thierry Aubin

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ABSTRACT. This book provides an introduction to differential geometry, with prinicpal emphasis on Riemannian geometry. It covers the essentials, concluding with a chapter on the Yamabe problem, which shows what research in the field looks like. It is a textbook, at a level which is accessible to graduate students. Its aim is to facilitate the study and the teaching of differential geometry. It is teachable on a chapter-by-chapter basis. Many problems and a number of solutions are included; most of them extend the course itself, which is confined to the main topics, such as: differential manifolds, submanifolds, differential mappings, tangent vectors, differential forms, orientation, manifolds with boundary, Lie derivative, integration of p-direction field, connection, torsion, curvature, geodesics, covariant derivative, Riemannian manifolds, exponential mapping, and spectrum.

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A mon Professeur
André Lichnerowicz
(in memoriam)

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Preface

This book provides an introduction to differential geometry, with principal emphasis on Riemannian geometry. It can be used as a course for second-year graduate students. The main theorems are presented in complete detail, but the student is expected to provide the details of certain arguments. We assume that the reader has a good working knowledge of multidimensional calculus and point-set topology.

Many readers have been exposed to the elementary theory of curves and surfaces in three-space, including tangent lines and tangent planes. But these techniques are not necessary prerequisites for this book.

In this book we work abstractly, so that the notion of tangent space does not necessarily have a concrete realization. Nevertheless we will eventually prove Whitney's theorem asserting that any abstract n-dimensional manifold may be imbedded in the Euclidean space $\mathbb{R}^p$ if p is sufficiently large.

In order to develop the abstract theory, one must work hard at the beginning, to develop the notion of local charts, change of charts, and atlases. Once these notions are understood, the subsequent proofs are much easier, allowing one to obtain great generality with maximum efficiency. For example, the proof of Stokes' theorem—which is difficult in a concrete context—becomes transparent in the abstract context, reducing to the computation of the integral of a derivative of a function on a closed interval of the real line.

In Chapter I we find the first definitions and two important theorems, those of Whitney and Sard.

Chapter II deals with vector fields and differential forms.

Chapter III concerns integration of vector fields, then extends to p-plane fields. We cite in particular the interesting proof of the Frobenius theorem, which proceeds by mathematical induction on the dimension.

Chapter IV deals with connections, the most difficult notion in differential geometry. In Euclidean space the notion of parallel transport is intuitive, but on a manifold it needs to be developed, since tangent vectors at distinct points are not obviously related. Loosely speaking, a connection defines an infinitesimal direction of motion in the tangent bundle, or, equivalently, a connection defines a sort of directional derivative of a vector field with respect to another vector. This concrete notion of connection is rarely taught in books on connections. In our work we devote ten pages to developing these ideas, together with the related notions of torsion, curvature and a working knowledge of the covariant derivative. All of these notions are essential to the study of real or complex manifolds.

In Chapter V we specialize to Riemannian manifolds. The viewpoint here is to deduce global properties of the manifold from local properties of curvature, the final goal being to determine the manifold completely.

In Chapter VI we explore some problems in partial differential equations which are suggested by the geometry of manifolds.

The last three chapters are devoted to global notation, specifically to using the covariant derivative instead of computing in local coordinates with partial derivatives. In some cases we are able to reduce a page of computation in local coordinates to just a few lines of global computation. We hope to further encourage the use of global notation among differential geometers.

The aim of this book is to facilitate the teaching of differential geometry. This material is useful in other fields of mathematics, such as partial differential equations, to name one. We feel that workers in PDE would be more comfortable with the covariant derivative if they had studied it in a course such as the present one. Given that this material is rarely taught, one may ask why? We feel that it requires a substantial amount of effort, and there is a shortage of good references. Of course there are reference books such as Kobayashi and Nomizu [5], which can be consulted for specific information, but that book is not written as a text for students of the subject.

The present book is made to be teachable on a chapter-by-chapter basis, including the solution of the exercises. The exercises are of varying difficulty, some being straightforward or solved in existing literature; others are more challenging and more directly related to our approach.

This book is an outgrowth of a course which I presented at the Université Paris VI. I have included many problems and a number of solutions. Some of these originated from examinations in the course. I am very grateful to my friend Mark Pinsky, who agreed to read the manuscript from beginning

to end. His comments allowed me to make many improvements, especially in the English. I would like to thank also one of my students, Sophie Bismuth, who helped me to prepare the final draft of this book.

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Subject Index

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Notation

Basic Notation

We use the Einstein summation convention.
Positive means strictly positive.
Nonnegative means positive or zero.
Compact manifold means compact manifold without boundary unless we say otherwise.
$\mathbb{N}$ is the set of positive integers, $n \in \mathbb{N}$.
$\mathbb{R}^n$ is the Euclidean n-space, $n \geq 2$, with points $x = (x^1, x^2, ..., x^n)$, $x^i \in \mathbb{R}$, the set of real numbers.
When it is not otherwise stated, a coordinate system $\{x^i\}_{1 \leq i \leq n}$ in $\mathbb{R}^n$ (or (x, y, z, t) in $\mathbb{R}^4$) is chosen to be orthonormal.
We often write ∂_i for $\partial/\partial x_i$ and ∂_{ij} for $\partial_i \partial_j$.
Sometimes we write ∇_{ij} for $\nabla_i \nabla_j$.
$]a, b[$ or (a, b) means an open interval in $\mathbb{R}$. (a, b) may also be the point of $\mathbb{R}^2$ whose coordinates are a and b.

Notation Index

$T_P(M)$ 2.3
$T(M)$ 2.5
$T^\star(M)$ 2.5
$T(X,Y)$ 4.6
$T(n,p)$ 1.20
$T(n,p,k)$ 1.21
$T_r^s(M)$ 2.14
$\Gamma(M)$ space of vector fields on M. 2.14
Γ_{ij}^k Christoffel symbols. 4.3, 4.5, 5.5
Δ Laplacian operator. 5.18
∂M boundary of M. 2.35, 2.36
δ codifferential. 5.17
δ_j^k Kronecker's symbol. 0.14, 5.5
η 5.15
$\Lambda^p(M)$ 2.14
$\Lambda(M)$ 2.22
μ Lebesgue measure. 0.28
$\sigma(X,Y)$ sectional curvature. 5.9
$\Phi_\star$ 2.6
$\Phi^\star$ 2.7, 2.23
χ the Euler-Poincaré characteristic. 5.22
ω_n the volume of the unit sphere S_n
∇u 4.13
$\nabla Y = \nabla_i Y^j dx^i \otimes \partial/\partial x^j$ 4.4
$(\partial/\partial x^i)_P$ tangent vector at P. 2.3
$[X,Y]$ bracket. 2.15
$\star$ adjoint operator. 5.1
$\otimes$ tensor product. 0.13

This textbook for graduate students is intended as an introduction to differential geometry with principal emphasis on Riemannian geometry. Chapter I explains basic definitions and gives the proofs of the important theorems of Whitney and Sard. Chapter II deals with vector fields and differential forms. Chapter III addresses integration of vector fields and p-plane fields. Chapter IV develops the notion of connection on a Riemannian manifold considered as a means to define parallel transport on the manifold. The author also discusses related notions of torsion and curvature, and gives a working knowledge of the covariant derivative. Chapter V specializes on Riemannian manifolds by deducing global properties from local properties of curvature, the final goal being to determine the manifold completely. Chapter VI explores some problems in PDEs suggested by the geometry of manifolds.

The author is well known for his significant contributions to the field of geometry and PDEs—particularly for his work on the Yamabe problem—and for his expository accounts on the subject.

The text contains many problems and solutions, permitting the reader to apply the theorems and to see concrete developments of the abstract theory.

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