Fourier Analysis

Javier Duoandikoetxea

Translated and revised by
David Cruz-Uribe, SFO

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by Javier Duoandikoetxea Zuazo

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ABSTRACT. The purpose of this book is to develop Fourier analysis using the real variable methods introduced by A. P. Calderón and A. Zygmund. It begins by reviewing the theory of Fourier series and integrals, and introduces the Hardy-Littlewood maximal function. It then treats the Hilbert transform and its higher dimensional analogues, singular integrals. In subsequent chapters it discusses some more recent topics: $H^1$ and $BMO$, weighted norm inequalities, Littlewood-Paley theory, and the $T1$ theorem. At the end of each chapter are extensive references and notes on additional results.

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10 9 8 7 6 5 4 3 2 14 13 12 11 10 09
Dedicated to the memory of
José Luis Rubio de Francia, my teacher and friend,
who would have written a much better book than I have
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Preface

Fourier Analysis is a large branch of mathematics whose point of departure is the study of Fourier series and integrals. However, it encompasses a variety of perspectives and techniques, and so many different introductions with that title are possible. The goal of this book is to study the real variable methods introduced into Fourier analysis by A. P. Calderón and A. Zygmund in the 1950’s.

We begin in Chapter 1 with a review of Fourier series and integrals, and then in Chapters 2 and 3 we introduce two operators which are basic to the field: the Hardy-Littlewood maximal function and the Hilbert transform. Even though they appeared before the techniques of Calderón and Zygmund, we treat these operators from their point of view. The goal of these techniques is to enable the study of analogs of the Hilbert transform in higher dimensions; these are of great interest in applications. Such operators are known as singular integrals and are discussed in Chapters 4 and 5 along with their modern generalizations. We next consider two of the many contributions to the field which appeared in the 1970’s. In Chapter 6 we study the relationship between $H^1$, $BMO$ and singular integrals, and in Chapter 7 we present the elementary theory of weighted norm inequalities. In Chapter 8 we discuss Littlewood-Paley theory; its origins date back to the 1930’s, but it has had extensive later development which includes a number of applications. Those presented in this chapter are useful in the study of Fourier multipliers, which also uses the theory of weighted inequalities. We end the book with an important result of the 80’s, the so-called $T1$ theorem, which has been of crucial importance to the field.

At the end of each chapter there is a section in which we try to give some idea of further results which are not discussed in the text, and give
references for the interested reader. A number of books and all the articles cited appear only in these notes; the bibliography at the end of the text is reserved for books which treat in depth the ideas we have presented.

The material in this book comes from a graduate course taught at the Universidad Autónoma de Madrid during the academic year 1988-89. Part of it is based on notes I took as a student in a course taught by José Luis Rubio de Francia at the same university in the fall of 1985. It seemed to have been his intention to write up his course, but he was prevented from doing so by his untimely death. Therefore, I have taken the liberty of using his ideas, which I learned both in his class and in many pleasant conversations in the hallway and at the blackboard, to write this book. Although it is dedicated to his memory, I almost regard it as a joint work. Also, I would like to thank my friends at the Universidad Autónoma de Madrid who encouraged me to teach this course and to write this book.

The book was first published in Spanish in the Colección de Estudios of the Universidad Autónoma de Madrid (1991), and then was republished with only some minor typographical corrections in a joint edition of Addison-Wesley/Universidad Autónoma de Madrid (1995). From the very beginning some colleagues suggested that there would be interest in an English translation which I never did. But when Professor David Cruz-Uribe offered to translate the book I immediately accepted. I realized at once that the text could not remain the same because some of the many developments of the last decade had to be included in the informative sections closing each chapter together with a few topics omitted from the first edition. As a consequence, although only minor changes have been introduced to the core of the book, the sections named “Notes and further results” have been considerably expanded to incorporate new topics, results and references.

The task of updating the book would have not been accomplished as it has been without the invaluable contribution of Professor Cruz-Uribe. Apart from reading the text, suggesting changes and clarifying obscure points, he did a great work on expanding the above mentioned notes, finding references and proposing new results to be included. The improvements of this book with respect to the original have certainly been the fruit of our joint work, and I am very grateful to him for sharing with me his knowledge of the subject much beyond the duties of a mere translator.

Javier Duoandikoetxea

Bilbao, June 2000
Acknowledgment: The translator would like to thank the Ford Foundation and the Dean of Faculty at Trinity College for their generous support during the academic year 1998–99. It was during this year-long sabbatical that this project was conceived and the first draft of the translation produced.
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Preliminaries

Here we review some notation and basic results, but we assume that they are mostly well known to the reader. For more information, see, for example, Rudin [14].

In general we will work in \( \mathbb{R}^n \). The Euclidean norm will be denoted by \(| \cdot |\). If \( x \in \mathbb{R}^n \) and \( r > 0 \),

\[
B(x, r) = \{ y \in \mathbb{R}^n : |x - y| < r \}
\]

is the ball with center \( x \) and radius \( r \). Lebesgue measure in \( \mathbb{R}^n \) is denoted by \( dx \) and on the unit sphere \( S^{n-1} \) in \( \mathbb{R}^n \) by \( d\sigma \). If \( E \) is a subset of \( \mathbb{R}^n \), \( |E| \) denotes its Lebesgue measure and \( \chi_E \) its characteristic function: \( \chi_E(x) = 1 \) if \( x \in E \) and 0 if \( x \notin E \). The expressions \textit{almost everywhere} or \textit{for almost every} \( x \) refer to properties which hold except on a set of measure 0; they are abbreviated by “a.e.” and “a.e. \( x \).”

If \( a = (a_1, \ldots, a_n) \in \mathbb{N}^n \) is a multi-index and \( f : \mathbb{R}^n \to \mathbb{C} \), then

\[
D^a f = \frac{\partial^{|a|} f}{\partial x_1^{a_1} \cdots \partial x_n^{a_n}},
\]

where \( |a| = a_1 + \cdots + a_n \) and \( x^a = x_1^{a_1} \cdots x_n^{a_n} \).

Let \( (X, \mu) \) be a measure space. \( L^p(X, \mu) \), \( 1 \leq p < \infty \), denotes the Banach space of functions from \( X \) to \( \mathbb{C} \) whose \( p \)-th powers are integrable; the norm of \( f \in L^p(X, \mu) \) is

\[
\|f\|_p = \left( \int_X |f|^p \, d\mu \right)^{1/p}
\]
$L^\infty(X, \mu)$ denotes the Banach space of essentially bounded functions from $X$ to $\mathbb{C}$; more precisely, functions $f$ such that for some $C > 0$,

$$\mu(\{x \in X : |f(x)| > C\}) = 0.$$ 

The norm of $f$, $\|f\|_{\infty}$, is the infimum of the constants with this property. In general $X$ will be $\mathbb{R}^n$ (or a subset of $\mathbb{R}^n$) and $d\mu = dx$; in this case we often do not give the measure or the space but instead simply write $L^p$. For general measure spaces we will frequently write $L^p(X)$ instead of $L^p(X, \mu)$; if $\mu$ is absolutely continuous and $d\mu = w \, dx$ we will write $L^p(w)$. The conjugate exponent of $p$ is always denoted by $p'$:

$$\frac{1}{p} + \frac{1}{p'} = 1.$$ 

The triangle inequality on $L^p$ has an integral version which we refer to as Minkowski’s integral inequality and which we will use repeatedly. Given measure spaces $(X, \mu)$ and $(Y, \nu)$ with $\sigma$-finite measures, the inequality is

$$\left( \int_X \left( \int_Y |f(x, y)|^p \, d\nu(y) \right)^{\frac{1}{p}} \, d\mu(x) \right)^{\frac{1}{p}} \leq \int_Y \left( \int_X |f(x, y)|^p \, d\mu(x) \right)^{\frac{1}{p}} \, d\nu(y).$$

The convolution of two functions $f$ and $g$ defined on $\mathbb{R}^n$ is given by

$$f \ast g(x) = \int_{\mathbb{R}^n} f(y) g(x - y) \, dy = \int_{\mathbb{R}^n} f(x - y) g(y) \, dy$$

whenever this expression makes sense.

The spaces of test functions are $C^\infty_c(\mathbb{R}^n)$, the space of infinitely differentiable functions of compact support, and $\mathcal{S}(\mathbb{R}^n)$, the so-called Schwartz functions. A Schwartz function is an infinitely differentiable function which decreases rapidly at infinity (more precisely, the function and all its derivatives decrease more rapidly than any polynomial increases). Given the appropriate topologies, their duals are the spaces of distributions and tempered distributions. It makes sense to define the convolution of a distribution and a test function as follows: if $T \in C^\infty_c(\mathbb{R}^n)'$ and $f \in C^\infty_c(\mathbb{R}^n)$, then

$$T \ast f(x) = \langle T, \tau_x \hat{f} \rangle,$$

where $\hat{f}(y) = f(-y)$ and $\tau_x f(y) = f(x + y)$. Note that this definition coincides with the previous one if $T$ is a locally integrable function. Similarly, we can take $T \in \mathcal{S}(\mathbb{R}^n)'$ and $f \in \mathcal{S}(\mathbb{R}^n)$. We denote the duality by either $\langle T, f \rangle$ or $T(f)$ without distinction.

References in square brackets are to items in the bibliography at the end of the book.

Finally, we remark that $C$ will denote a positive constant which may be different even in a single chain of inequalities.
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Fourier analysis encompasses a variety of perspectives and techniques. This volume presents the real variable methods of Fourier analysis introduced by Calderón and Zygmund. The text was born from a graduate course taught at the Universidad Autónoma de Madrid and incorporates lecture notes from a course taught by José Luis Rubio de Francia at the same university.

Motivated by the study of Fourier series and integrals, classical topics are introduced, such as the Hardy-Littlewood maximal function and the Hilbert transform. The remaining portions of the text are devoted to the study of singular integral operators and multipliers. Both classical aspects of the theory and more recent developments, such as weighted inequalities, $H^1$, $BMO$ spaces, and the $T_1$ theorem, are discussed.

Chapter 1 presents a review of Fourier series and integrals; Chapters 2 and 3 introduce two operators that are basic to the field: the Hardy-Littlewood maximal function and the Hilbert transform. Chapters 4 and 5 discuss singular integrals, including modern generalizations. Chapter 6 studies the relationship between $H^1$, $BMO$, and singular integrals; Chapter 7 presents the elementary theory of weighted norm inequalities. Chapter 8 discusses Littlewood-Paley theory, which had developments that resulted in a number of applications. The final chapter concludes with an important result, the $T_1$ theorem, which has been of crucial importance in the field.

This volume has been updated and translated from the Spanish edition that was published in 1995. Minor changes have been made to the core of the book; however, the sections, “Notes and Further Results” have been considerably expanded and incorporate new topics, results, and references. It is geared toward graduate students seeking a concise introduction to the main aspects of the classical theory of singular operators and multipliers. Prerequisites include basic knowledge in Lebesgue integrals and functional analysis.