Stochastic Analysis on Manifolds

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ABSTRACT. This book is intended for advanced graduate students and researchers who are familiar with both euclidean stochastic analysis and basic differential geometry. It begins with a brief review of stochastic differential equations on euclidean space. After presenting the basics of stochastic analysis on manifolds, we introduce Brownian motion on a Riemannian manifold and study the effect of curvature on its behavior. We then show how to apply Brownian motion to geometric problems and vice versa by several typical examples, including eigenvalue estimates and probabilistic proofs of the Gauss-Bonnet-Chern theorem and the Atiyah-Singer index theorem for Dirac operators. The book ends with an introduction to stochastic analysis on the path space over a Riemannian manifold, a topic of much current research interest.

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In Memory of My Beloved Mother
Zhu Pei-Ru (1926-1996)

Quält mich Erinnerung, daß ich verübet,
So manche Tat, die dir das Herz betrübet?
Das schöne Herz, das mich so sehr geliebet?
....
Doch da bist du entgegen mir gekommen,
Und ach! was da in deinem Aug geschwommen,
Das war die süße, langgesuchte Liebe.

Heinrich Heine, *An Meine Mutter.*

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Preface

A large part of modern analysis is centered around the Laplace operator and its generalizations in various settings. On the other hand, under certain technical conditions one can show that the generator of a continuous strong Markov process must be a second order elliptic operator. These are the two facts that make the connection between probability and analysis seem natural. If we adopt the view that Brownian paths are the "characteristic lines" for the Laplace operator, then it is no longer surprising that solutions of many problems associated with the Laplace operator can be explicitly represented by Brownian motion, for it is a well known fact in the theory of partial differential equations that explicit solutions are possible if one knows characteristic lines. While analysts are interested in what happens in average, to a probabilist things usually happen at the path level. For these reasons probability theory, and Brownian motion in particular, has become a convenient language and useful tool in many areas of analysis. The purpose of this book is to explore this connection between Brownian motion and analysis in the area of differential geometry.

Unlike many time-honored areas of mathematics, stochastic analysis has neither a well developed core nor a clearly defined boundary. For this reason the choice of the topics in this book reflects heavily my own interest in the subject; its scope is therefore much narrower than the title indicates. My purpose is to show how stochastic analysis and differential geometry can work together towards their mutual benefit. The book is written mainly from a probabilist's point of view and requires for its understanding a solid background in basic euclidean stochastic analysis. Although necessary geometric facts are reviewed throughout the book, a good knowledge of differential geometry is assumed on the part of the reader. Because of its somewhat unusual dual prerequisites, the book is best suited for highly motivated advanced graduate students in either stochastic analysis or differential geometry and for researchers in these and related areas of mathematics. Notably absent from the book are a collection of exercises commonly associated with books of this kind, but throughout the book there are many proofs which are nothing but an invitation to test the reader's understanding of the topics under discussion.

During the writing of this book, I have greatly benefited from several existant monographs on the subject; these include:

- N. Ikeda and S. Watanabe: *Stochastic Differential Equations and Diffusion Processes*, 2nd edition, North-Holland/Kodansha (1989);
- K. D. Elworthy: *Stochastic Differential Equations on Manifolds*, Cambridge University Press (1982);
- M. Emery: *Stochastic Calculus in Manifolds*, Springer (1989);
- P. Malliavin: *Stochastic Analysis*, Springer (1997).

Overlaps with these works are not significant, and they and the more recent *An Introduction to the Analysis of Paths on a Riemannian Manifold*, American Mathematical Society (2000), by D. W. Stroock, are warmly recommended to the reader.

This book could not have been written without constant support from my wife, who has taken more than her fair share of family duties during its long gestation period. I would like to take this opportunity to thank Elena Kosygina, Tianhong Li, Banghe Li, and Mark A. Pinsky for reading early drafts of various parts of the book and for their valuable suggestions. I would also like to acknowledge many years of financial support through research grants from the National Science Foundation. Most of the book is based on the lectures I have delivered at various places during the late 1990s, notably at Academica Sinica in Beijing (1995), IAS/Park City Mathematics Institute in Princeton (1996), Institut Henri Poincaré in Paris (1998), and Northwestern University in Evanston (1999), and I would like to thank my audiences for their comments and suggestions.

Elton P. Hsu
Hinsdale, IL
October, 2001.

General Notations

Notation	Definition
$\mathscr{B}(X)$	Borel σ-field of a metric space X
$\mathscr{B}_*$	Borel filtration of $W(M) = \{\mathscr{B}_t\}$
$B(x; R)$	(geodesic) ball of radius R centered at x
$\square_M$	Hodge-de Rham Laplacian on M
$\langle X, Y\rangle$	co-variation of semimartingales X and Y
$\langle X\rangle$	quadratic variation of $X = \langle X, X\rangle$
$C^\infty(M)$	smooth functions on M
C_x	cutlocus of x
$\circ$	Stratonovich stochastic integral, $X \circ dY = XdY + \frac{1}{2}d\langle X, Y\rangle$
$c(X)$	Clifford multiplication by X (in CHAPTER 7)
d	exterior differentiation
$d_M(x, y)$	distance between x and y on M
D	gradient operator on $P_o(M)$
D	Dirac operator (in CHAPTER 7)
D_h	Cameron-Martin vector field on $P_o(M)$
δ	dual of exterior differentiation
Δ_M	Laplace-Beltrami operator on $M = -(d\delta + \delta d)$
$\Delta_{\mathscr{O}(M)}$	Bochner's horizontal Laplacian on $\mathscr{O}(M) = \sum_{i=1}^d H_i^2$
A^*	dual operator of A
$e(\omega)$	lifetime (explosion time) of a path ω
$\mathrm{End}(V)$	space of linear transforms on V
$\exp_o$	exponential map based at o
$\mathscr{F}_*$	filtration of σ-fields $= \{\mathscr{F}_t\}$

Notation	Definition
$\mathscr{F}(M)$	frame bundle of M
$\mathscr{F}_*^X$	filtration generated by process $X = \{\mathscr{F}_t^X\}$
$\Gamma(E)$	space of sections of vector bundle E
Γ_{ij}^k	Christoffel symbols
$\Gamma(f, g)$	Γ of f and $g = L(fg) - fLg - gLf$
$GL(d, \mathbb{R})$	set of real nonsingular $(d \times d)$ matrices
H_i	fundamental horizontal vector field on $\mathscr{F}(M)$
$\mathscr{H}$	Cameron-Martin space
$i(X)$	interior product with X
i_x	injectivity radius at x
i_K	injectivity radius on $K = \min\{i_x : x \in K\}$
$I(J, J)$	index form of a vector field J
$K_M(x)$	set of sectional curvatures at a point $x \in M$
L	Orstein-Uhlenbeck operator (in CHAPTER 8)
$\mathscr{M}(d, l)$	space of $(d \times l)$ matrices
$M^\dagger$	transpose of a matrix M
$\widehat{M}$	one-point compactification of a manifold M $=M \cup \{\partial_M\}$
$\lvert \cdot \rvert_{\mathscr{H}}$	Cameron-Martin norm
∇	connection and covariant differentiation
$\nabla^2 f$	Hessian of f
$\nabla^H G$	horizontal gradient of $G = \{H_1 G, \ldots, H_d G\}$
$O(d)$	$(d \times d)$ orthogonal group
$\mathfrak{o}(d)$	$(d \times d)$ anti-symmetric matrices
$\mathscr{O}(M)$	orthonormal frame bundle of M
Ω	curvature form
$\mathrm{Pf}(A)$	Pfaffian of an anti-symmetric matrix A
$P_o(M)$	space of paths on M starting from o with time length 1
$\mathbb{P}_x$	law of Brownian motion starting from x
$\mathbb{P}_{x,y;t}$	law of Brownian bridge from x to y with time length t
$p_M(t, x, y)$	heat kernel on a Riemannian manifold M
$\{P_t\}$	heat semigroup= $e^{t\Delta_M/2}$
$\{\mathscr{P}_t\}$	Ornstein-Uhlenbeck semigroup= $e^{tL/2}$
Π	second fundamental form
$\mathbb{R}^N$	euclidean space of dimension N
$\mathbb{R}_+$	set of nonnegative real numbers $= [0, \infty)$,
$R(X, Y)Z$	curvature tensor evaluated at X, Y, Z
$\mathrm{Ric}_M(x)$	set of Ricci curvatures at a point $x \in M$

Notation	Definition
Ric_u	Ricci transform at a frame $u \in \mathscr{O}(M)$
$\mathscr{S}_+(d)$	$(d \times d)$ symmetric positive definite matrices
$\mathbb{S}^n$	n-sphere
$\mathrm{Spin}(d)$	Spin group
$\mathscr{S}(M)$	spin bundle over a spin manifold M
$\mathscr{S}\mathscr{P}(M)$	$\mathrm{Spin}(d)$-principal bundle over M
X^*	horizontal lift of $X \in TM$ to the frame bundle $\mathscr{F}(M)$
τ_D	first exit time of $D = \inf\{t : X_t \notin D\}$
θ_t	shift operator in a path space: $(\theta_t\omega)_s = \omega_{t+s}$
T_K	first hitting time of $K = \inf\{t : X_t \in K\}$
$\widetilde{\theta}$	scalarization of a tensor θ
Trace	supertrace (in CHAPTER 7)
TM	tangent bundle of a manifold M
T_xM	tangent space of M at x
T_x^*M	cotangent space of M at x
$\wedge_x^p M$	space of p-forms at a point $x \in M$
$W(M)$	space of paths on M with lifetimes

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