Introduction to the Theory of Differential Inclusions

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Abstract. Differential inclusions theory is presented at an elementary level. The emphasis is given to applications such as viability theory, controllability, necessary conditions of optimality, asymptotic stability at first approximation, and the stabilization problem. The text is intended for graduate students who specialize in pure and applied analysis.

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Preface

The aim of this text is to provide an introductory treatment of the theory of differential inclusions that will be accessible to anyone having a basic knowledge of analysis, theory of functions, and ordinary differential equations.

The present book is an expanded record of lectures given at the International School for Advanced Studies (SISSA), Trieste, Italy, and at the Universities of Évora and Porto, Portugal, during the last few years.

Most of the material in this text was written when the author was visiting SISSA. I acknowledge the opportunity offered by the Functional Analysis Sector of SISSA to work there. I wish to express my thanks to Boris Mordukhovich, who read the draft of the manuscript and made important remarks. I also thank the students who made suggestions that have improved the presentation.

Georgi Smirnov

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Introduction

In this text we consider differential inclusions

$$\dot{x} \in F(x),$$

where F is a set-valued map which associates with any point $x \in R^n$ a set $F(x) \subset R^n$. Differential inclusions serve as models for many dynamical systems. Obviously, any process described by an ordinary differential equation

$$\dot{x} = f(x)$$

can be described by a differential inclusion with the right-hand side $F(x) = \{f(x)\}$ as well. A system of differential inequalities

$$\dot{x}^i \leq f^i(x^1, \dots, x^n),\ i = \overline{1, n}$$

can also be considered as a differential inclusion. If an implicit differential equation

$$f(x, \dot{x}) = 0$$

is given, then we can put $F(x) = \{v \mid f(x, v) = 0\}$ to reduce it to a differential inclusion. Differential inclusions are used to study ordinary differential equations with an inaccurately known right-hand side. Suppose that the right-hand side of a differential equation is in an ϵ-neighborhood of a given function $f(x)$. Then any solution of the differential equation is a solution to the differential inclusion

$$\dot{x} \in f(x) + \epsilon B_n,$$

where B_n is a unit ball in R^n centered at zero.

Differential inclusions play a crucial role in the theory of differential equations with a discontinuous right-hand side. The investigation of such equations is of great importance since they model the performance of various mechanical and electrical devices as well as the behavior of automatic control systems. Differential equation

$$\dot{x} = f(x)$$

with discontinuous f is a rather unpleasant object from the mathematical point of view. In particular it is impossible to prove existence theorems. However if solutions of the differential equation with discontinuous right-hand side are regarded to be solutions to the differential inclusion

$$\dot{x} \in \bigcap_{\epsilon>0} \operatorname{cl} \operatorname{co} f(x + \epsilon B_n),$$

then it is possible to develop a rigorous mathematical theory of discontinuous systems.

One of the most important examples of differential inclusions comes from control theory. Consider a control system

$$\dot{x} = f(x, u), \ u \in U,$$

where u is a control parameter. It appears that the control system and the differential inclusion

$$\dot{x} \in f(x, U) = \bigcup_{u \in U} f(x, u)$$

have the same trajectories. If the set of controls depends on x, that is, $U = U(x)$, then we obtain the differential inclusion

$$\dot{x} \in f(x, U(x)).$$

The equivalence between a control system and the corresponding differential inclusion is the central idea used to prove existence theorems in optimal control theory.

Since the dynamics of economical, social, and biological macrosystems is multivalued, differential inclusions serve as natural models in macrosystem dynamics. Differential inclusions are also used to describe some systems with hysteresis.

Differential inclusion is a generalization of the notion of an ordinary differential equation. Therefore all problems considered for differential equations, that is, existence of solutions, continuation of solutions, dependence on initial conditions and parameters, are present in the theory of differential inclusions. Since a differential inclusion usually has many solutions starting at a given point, new issues appear, such as investigation of topological properties of the set of solutions, selection of solutions with given properties, evaluation of the reachability sets, etc. To solve the above problems special mathematical techniques were developed.

Thus, the differential inclusions are not only models for many dynamical processes but they also provide a powerful tool for various branches of mathematical analysis. As we have mentioned the differential inclusion techniques are applied to prove existence theorems in optimal control theory. They are used to derive sufficient conditions of optimality, play an essential role in the theory of control under conditions of uncertainty and in differential games theory.

In this text we discuss many important problems of differential inclusions theory. For simplicity of presentation we consider only differential inclusions with a convex-valued right-hand side. The mathematical difficulties appearing in the nonconvex case and outstanding techniques developed to overcome them are outside of the frame of this text. This assumption enables us to apply approximation techniques in order to solve many interesting problems. Namely, we approximate differential inclusions under consideration by differential (or even discrete-time) inclusions with simpler structure. We then solve the problem for the simplified dynamical systems and obtain the result for the original problem using a limiting procedure. This is the central idea of many proofs in the text that allows us to avoid complicated mathematical constructions. Note that this method simplifies the proofs of some classical theorems from the theory of ordinary differential equations (the Hukuhara theorem, the Kneser theorem, for instance).

The required mathematical background is knowledge of ordinary differential equations theory, the theory of functions, and functional analysis at an elementary level. The text is intended for graduate students who specialize in pure and applied analysis.

The text is organized as follows. The first chapter contains a brief introduction to convex analysis. In the second chapter we consider set-valued maps. The third chapter is devoted to the Mordukhovich version of nonsmooth analysis. Chapter 4 contains the main existence theorems and gives an idea of the approximation techniques used throughout the text. Chapter 5 is devoted to the viability problem, that is, the problem of selection of a solution to a differential inclusion which is contained in a given set. The controllability problem is considered in Chapter 6. In Chapter 7 we discuss extremal problems for differential inclusions. Stability theory for differential inclusions is presented in Chapter 8. In the last chapter we deal with the stabilization problem.

At the end of the book we suggest further reading intended to provide more detailed coverage of the relevant topics, but no attempt is made to survey the literature on the subject.

Comments

In these bibliographical comments we do not review all works on differential inclusions. We mention only books devoted to the theory of differential inclusions or its applications, fundamental papers that are interesting from the historical point of view and show the development of the theory, and some papers closely connected with the material of the text. The works are cited in chronological order.

Introduction. The theory of differential inclusions appeared in 30's when Marchaud [**77**] and Zaremba [**128**] proved the first existence theorems. However at that time there were no applications of the results obtained, and mathematicians did not pay much attention to the subject. At the end of 50's optimal control theory was developed (see Bellman [**16**], Pontryagin, Boltyanski, Gamkrelidze, and Mischenko [**95**]). It was a great impetus for differential inclusions theory. Filippov [**41**] and Wazewski [**125**] established the correspondence between control systems and differential inclusions and derived existence results for optimal control problems. Another motivation came from the theory of differential equations with discontinuous right-hand side. Filippov [**42**] suggested to define solutions of such systems as solutions to differential inclusions. Since the early 60's the theory of differential inclusions can be considered as an independent branch of mathematics with its own problems, methods, and applications.

There exist many monographs devoted to various aspects of differential inclusions theory. In Aubin and Cellina [**9**] the emphasis is given to selection problems, existence theorems, and viability theory. The theory of differential equations with discontinuous right-hand side is considered in Filippov [**45**]. In Tolstonogov [**119**] differential inclusions in Banach space are studied.

The book of Deimling [**40**] contains many results concerning the existence of solutions and qualitative properties of solution sets.

The discussion of the role of differential inclusions in economics, sociology, and biology can be found in Aubin [**8**], where a complete presentation of viability theory is given. Extremal problems for differential inclusions are considered in Pshenichnyi [**98**], Clarke [**34**], Blagodatskikh and Filippov [**18**], Mordukhovich [**83**], and Kisielewicz [**63**]. Differential inclusions are applied to differential games theory in Krasovski and Subbotin [**66**] and Aubin [**8**], and to the theory of control under conditions of uncertainty in Kurzhanski [**69**]. Krasnoselski and Pokrovski [**65**] used differential inclusions to describe systems with hysteresis. Applications in mechanics are considered in Monteiro Marques [**80**].

More complete information on differential inclusions can be obtained from the bibliography contained in the mentioned books.

Chapter 1. There are many monographs on convex analysis. We mention only books by Rockafellar [**99**] and Pshenichnyi [**98**]. The solutions of the problems to this chapter also can be found there.

Chapter 2. The theory of set-valued maps is considered in Castaing and Valadier [**28**], Pshenichnyi [**98**], Aubin and Cellina [**9**], Aubin and Ekeland [**11**], Aubin and Frankowska [**12**]. The concept of directional derivative for set-valued maps was introduced by Aubin [**7**]. The approximation theorem from Section 3 belongs to Haddad and Lasri [**55**]. The first extension theorem for set-valued maps was proved by Cellina [**29**]. The decomposition of the compliment of a closed set into cubes used in Section 4 first appeared in Whitney [**126**]. Elementary proofs of the Brouwer fixed point theorem can be found in [**15**], [**120**]. The notion of a σ-selectionable map appeared in Haddad and Lasri [**55**]. Many other results on the selection problem can be found in Aubin and Cellina [**9**].

Convex processes were introduced by Rockafellar [**99**]. Other set-valued versions of Perron's theorem can be found in Aubin and Ekeland [**11**] and Aubin, Frankowska, and Olech [**13**]. The structure of convex processes was studied in Aubin, Frankowska, and Olech [**13**] and Smirnov [**106**].

Problems 4 and 5 are from Pontryagin [**93**].

Chapter 3. Nonsmooth calculus appeared at the end of the 60's. The first concepts are described in Ioffe and Tikhomirov [**60**] and Pshenichnyi [**98**], for example. A very successful version of nonsmooth analysis was developed by Clarke [**32**]. A complete presentation of this approach is given in Clarke [**34**].

Here we consider the version of nonsmooth analysis developed by Mordukhovich [**81**], [**82**], [**83**], [**84**], [**87**]. The separation theorem for nonconvex sets was proved by Kruger and Mordukhovich [**68**].

Related nonsmooth calculus appeared in Ioffe [**57**] and Rockafellar [**101**], Rockafellar and Wets [**103**]. An alternative approach can be found in Sussmann [**116**].

Problems 1, 2, and 6-10 are from Mordukhovich [**83**]. (Many of them are simplified versions.) The concept of tent was introduced by Boltyanski. Problems 3 and 4 go back to Rockafellar [**100**] and Watkins [**124**].

Chapter 4. The facts from the theory of functions and functional analysis gathered in Section 1 are proved in the textbook by Kolmogorov and Fomin [**64**], for example. The first version of the existence theorem combined with the Gronwall inequality was proved by Filippov [**43**]. The continuous version of Filippov's result presented here is taken from Polovinkin and Smirnov [**92**]. Continuous versions of the Filippov theorem for differential inclusions with nonconvex-valued right-hand side and some related results are obtained in Polovinkin and Smirnov [**92**], Cellina [**30**], Ornelas [**89**], Colombo, Fryszkowski, Rzezuchowski, and Staicu [**37**], Fryszkowski and Rzezuchowski [**52**], Staicu and Wu [**113**], and Smirnov [**108**].

The most general existence result for differential inclusions with convex-valued right-hand sides was obtained by Davy [**38**]. In the nonconvex case it is much more difficult to prove existence theorems. The first existence theorems in the nonconvex case were proved by Filippov [**43**] for Lipschitzian differential inclusions. Then in Filippov [**44**] the existence result for differential inclusions with continuous right-hand sides was obtained. A very general existence theorem which covers the results of Davy [**38**] and Filippov [**44**] was proved by Olech [**88**]. For differential inclusions with nonconvex lower semi-continuous right-hand sides, existence theorems were proved by Bressan [**20**] and Lojasiewicz [**74**]. A general existence result in Banach spaces was obtained by Bressan and Colombo [**26**].

Some uniqueness results are contained in Aubin and Cellina [**9**] and Cellina [**31**].

Qualitative properties of solution sets to differential inclusions with convex-valued right-hand sides, were considered by Davy [**38**] and the nonconvex case was studied by Bressan [**21**], [**22**], [**23**], [**24**], and De Blasi and Pianigiani [**39**]. Qualitative properties for Lipschitzian differential inclusions were largely studied in Bressan, Cellina, and Fryszkowski [**25**] and Smirnov [**108**].

The examples of discontinuous differential equations are taken from Andronov, Vitt, and Khaikin [**1**].

Many results on the existence of optimal trajectories are gathered in Kisielewicz [**63**].

The results on the differentiability with respect to initial conditions and estimates of tangent cones to solution sets appeared in Frankowska [**47**], Polovinkin and Smirnov [**91**], and Frankowska and Kaskosz [**50**].

The approximation result of Section 7 is taken from Smirnov [**107**]. Some other approximation theorems can be found in Pshenichnyi [**98**], Mordukhovich [**83**], and Wolenski [**127**].

Problem 2 is a special case of a more general result from Filippov [**43**]. Problems 4 and 5 are from Gamkrelidze [**53**]. Problem 6 in the frame of optimal control theory was considered by Sussmann [**114**]. Problem 8 is from Bushenkov and Smirnov [**27**].

Chapter 5. Different results concerning monotone solutions were obtained by Roxin [**105**], Krbec [**67**], Aubin, Cellina, and Nohel [**10**], and Aubin [**7**] (see also [**9**]). The viability theorem was proved in Haddad [**54**]. The material of Section 3 is taken from Clarke [**32**] and from Aubin and Cellina [**9**]. The existence theorem for periodic solutions was obtained by Haddad and Lasri [**55**]. Viability theory is presented in Aubin and Cellina [**9**] and Aubin [**8**].

There are many monographs on the theory of differential games. We mention only the books by Friedman [**51**] and Krasovski and Subbotin [**66**]. The approach presented here is due to Pontryagin [**93**], [**94**] and Pontryagin and Mischenko [**96**].

Problem 3 is due to Krasnoselski. Problem 4 is from Pontryagin [**93**]. Problem 5 is a simplified version of more general results from Pontryagin and Mischenko [**96**] (see also Friedman [**51**]).

Chapter 6. Controllability of convex processes was studied in Aubin, Frankowska, and Olech [**13**]. The result on continuous differentiability on initial conditions is taken from [**92**]. Controllability at first approximation was considered in Blagodatskikh [**17**], Frankowska [**46**], and Polovinkin and Smirnov [**92**]. Dual forms of controllability conditions can be found, for example, in Clarke [**34**], Polovinkin and Smirnov [**92**], Mordukhovich [**83**], Frankowska and Kaskosz [**50**], Sussmann [**115**], [**117**], and Tuan [**121**]. Controllability at high order approximation is considered in Frankowska [**49**] and Sussmann [**115**], [**117**], [**118**].

Problems 1 and 2 are from Aubin, Frankowska, and Olech [**13**]. The techniques needed to solve Problem 5 can be found in Ioffe and Tikhomirov [**60**]. Problems 6 and 7 are from Polovinkin and Smirnov [**91**]. Problem 8 is a special case of a more general result from Polovinkin and Smirnov [**92**].

Chapter 7. First results on necessary conditions of optimality for differential inclusions were obtained by Boltyanski [**19**], Clarke [**33**], and Pshenichnyi [**97**] (see also Pshenichnyi [**98**] and Clarke [**34**]). The method of discrete approximations goes back to Halkin [**56**], Pshenichnyi [**97**], [**98**], and Mordukhovich [**82**], [**83**], [**85**], [**86**]. The necessary conditions proved here are taken from Smirnov [**107**]. Some other approaches to the problem are considered in Frankowska [**48**], Kaskosz and Lojasiewicz [**61**], [**62**], Lojasiewicz, Plis, and Suarez [**75**], Polovinkin and Smirnov [**91**], [**92**], Aseev [**5**], [**6**], Ioffe [**58**], Ioffe and Rockafellar [**59**], Loewen and Rockafellar [**71**], [**72**], Loewen and Vinter [**73**], Pinho and Vinter [**90**], Rockafellar [**102**], Rowland and Vinter [**104**], Sussmann [**115**], [**117**], [**118**], and Vinter and Zheng [**122**]. The problem with state constraints is considered in Arutyunov and Aseev [**2**], Arutyunov, Aseev, and Blagodatskikh [**3**], and Vinter and Zheng [**123**].

The model of hydroplane considered in Section 3 is a modification of a model from Andronov, Vitt, and Khaikin [**1**].

The mechanical model used in Problem 4 is due to Clarke [**34**]. Problem 6 is from Polovinkin and Smirnov [**92**]. Problem 8 is from Pshenichnyi [**98**].

Chapter 8. The concepts of stability for differential inclusions were introduced in Roxin [**105**]. Stability of an equilibrium position is comprehensively studied in Filippov [**45**]. Lyapunov functions are used to investigate the weak stability problem in Roxin [**105**], Krbec [**67**], Aubin [**7**], Aubin and Cellina [**9**], and Aubin [**8**]. The construction of a quadratic Lyapunov function for a linear asymptotically stable system appeared in Lyapunov [**76**]. For the general case, converse Lyapunov theorems are proved in Arzarello and Bacciotti [**4**], Clarke, Ledyaev, and Stern [**36**], and Lin, Sontag, and Wang [**70**]. Asymptotic stability for linear-selectionable differential inclusions was studied by Molchanov and Pyatnitski [**78**], [**79**]. Many other results on stability and asymptotic stability can be found in Filippov [**45**]. The results on weak asymptotic stability of convex processes and weak asymptotic stability at first approximation are taken from Smirnov [**106**]. The case of periodic differential inclusions is considered in Smirnov [**110**].

Problems 4-7 are from Smirnov [**110**].

Chapter 9. Literature on the stabilization problem is reviewed in Bacciotti [**14**] and Sontag [**112**]. The main results of this chapter are taken from Smirnov [**106**]. Some other aspects concerning the approach considered here are contained in Bushenkov and Smirnov [**27**] and Smirnov [**111**], [**109**]. Another point of view on the relation between weak asymptotic stability and stabilizability can be found in Clarke, Ledyaev, Sontag, and Subbotin [**35**].

All problems are from Bushenkov and Smirnov [**27**].

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