Introduction to the Theory of Random Processes

N. V. Krylov

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ABSTRACT. These lecture notes concentrate on some general facts and ideas of the theory of stochastic processes. The main objects of study are the Wiener processes, the stationary processes, the infinitely divisible processes, and the Itô stochastic equations.

Although it is not possible to cover even a noticeable portion of the topics listed above in a short course, the author sincerely hopes that after having followed the material presented here the reader will have acquired a good understanding of what kind of results are available and what kind of techniques are used to obtain them.

These notes are intended for graduate students and scientists in mathematics, physics and engineering interested in the theory of random processes and its applications.

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Preface

For about ten years between 1973 and 1986 the author was delivering a one-year topics course "Random Processes" at the Department of Mechanics and Mathematics of Moscow State University. This topics course was obligatory for third-fourth year undergraduate students (about 20 years of age) with major in probability theory and its applications. With great sympathy I remember my first students in this course: M. Safonov, A. Veretennikov, S. Anulova, and L. Mikhailovskaya. During these years the contents of the course gradually evolved, simplifying and shortening to the shape which has been presented in two 83 and 73 page long rotaprint lecture notes published by Moscow State University in 1986 and 1987. In 1990 I emigrated to the USA and in 1998 got the opportunity to present parts of the same course as a one-quarter topics course in probability theory for graduate students at the University of Minnesota. I thus had the opportunity to test the course in the USA as well as on several generations of students in Russia. What the reader finds below is a somewhat extended version of my lectures and the recitations which went along with the lectures in Russia.

The theory of random processes is an extremely vast branch of mathematics which cannot be covered even in ten one-year topics courses with minimal intersection of contents. Therefore, the intent of this book is to get the reader acquainted only with some parts of the theory. The choice of these parts was mainly defined by the duration of the course and the author's taste and interests. However, there is no doubt that the ideas, facts, and techniques presented here will be useful if the reader decides to move on and study some other parts of the theory of random processes.

From the table of contents the reader can see that the main topics of the book are the Wiener process, stationary processes, infinitely divisible processes, and Itô integral and stochastic equations. Chapters 1 and 3 are devoted to some techniques needed in other chapters. In Chapter 1 we discuss some general facts from probability theory and stochastic processes from the point of view of probability measures on Polish spaces. The results of this chapter help construct the Wiener process by using Donsker's invariance principle. They also play an important role in other issues, for instance, in statistics of random processes. In Chapter 3 we present basics of discrete time martingales, which then are used in one way or another in all subsequent chapters. Another common feature of all chapters excluding Chapter 1 is that we use stochastic integration with respect to random orthogonal measures. In particular, we use it for spectral representation of trajectories of stationary processes and for proving that Gaussian stationary processes with rational spectral densities are components of solutions to stochastic equations. In the case of infinitely divisible processes, stochastic integration allows us to obtain a representation of trajectories through jump measures. Apart from this and from the obvious connection between the Wiener process and Itô's calculus, all other chapters are independent and can be read in any order.

The book is designed as a textbook. Therefore it does not contain any new theoretical material but rather a new compilation of some known facts, methods and ways of presenting the material. A relative novelty in Chapter 2 is viewing the Itô stochastic integral as a particular case of the integral of nonrandom functions against random orthogonal measures. In Chapter 6 we give two proofs of Itô's formula: one is more or less traditional and the other is based on using stochastic intervals. There are about 128 exercises in the book. About 41 of them are used in the main text and are marked with an asterisk. The bibliography contains some references we use in the lectures and which can also be recommended as a source of additional reading on the subjects presented here, deeper results, and further references.

The author is sincerely grateful to Wonjae Chang, Kyeong-Hun Kim, and Kijung Lee, who read parts of the book and pointed out many errors, to Dan Stroock for his friendly critisizm of the first draft, and to Naresh Jain for useful suggestions.

Nicolai Krylov
Minneapolis, January 2001

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