

Introduction to the Theory of Random Processes

N. V. Krylov

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ABSTRACT. These lecture notes concentrate on some general facts and ideas of the theory of stochastic processes. The main objects of study are the Wiener processes, the stationary processes, the infinitely divisible processes, and the Itô stochastic equations.

Although it is not possible to cover even a noticeable portion of the topics listed above in a short course, the author sincerely hopes that after having followed the material presented here the reader will have acquired a good understanding of what kind of results are available and what kind of techniques are used to obtain them.

These notes are intended for graduate students and scientists in mathematics, physics and engineering interested in the theory of random processes and its applications.

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Contents

Preface	xi
Chapter 1. Generalities	1
§1. Some selected topics from probability theory	1
§2. Some facts from measure theory on Polish spaces	5
§3. The notion of random process	14
§4. Continuous random processes	16
§5. Hints to exercises	25
Chapter 2. The Wiener Process	27
§1. Brownian motion and the Wiener process	27
§2. Some properties of the Wiener process	32
§3. Integration against random orthogonal measures	39
§4. The Wiener process on $[0, \infty)$	50
§5. Markov and strong Markov properties of the Wiener process	52
§6. Examples of applying the strong Markov property	57
§7. Itô stochastic integral	61
§8. The structure of Itô integrable functions	65
§9. Hints to exercises	69
Chapter 3. Martingales	71

§1. Conditional expectations	71
§2. Discrete time martingales	78
§3. Properties of martingales	81
§4. Limit theorems for martingales	87
§5. Hints to exercises	92
Chapter 4. Stationary Processes	95
§1. Simplest properties of second-order stationary processes	95
§2. Spectral decomposition of trajectories	101
§3. Ornstein-Uhlenbeck process	105
§4. Gaussian stationary processes with rational spectral densities	112
§5. Remarks about predicting Gaussian stationary processes with rational spectral densities	117
§6. Stationary processes and the Birkhoff-Khinchin theorem	119
§7. Hints to exercises	127
Chapter 5. Infinitely Divisible Processes	131
§1. Stochastically continuous processes with independent increments	131
§2. Lévy-Khinchin theorem	137
§3. Jump measures and their relation to Lévy measures	144
§4. Further comments on jump measures	154
§5. Representing infinitely divisible processes through jump measures	155
§6. Constructing infinitely divisible processes	160
§7. Hints to exercises	166
Chapter 6. Itô Stochastic Integral	169
§1. The classical definition	169
§2. Properties of the stochastic integral on H	174
§3. Defining the Itô integral if $\int_0^T f_s^2 ds < \infty$	179
§4. Itô integral with respect to a multidimensional Wiener process	186
§5. Itô's formula	188
§6. An alternative proof of Itô's formula	195

§7. Examples of applying Itô's formula	200
§8. Girsanov's theorem	204
§9. Stochastic Itô equations	211
§10. An example of a stochastic equation	216
§11. The Markov property of solutions of stochastic equations	220
§12. Hints to exercises	225
Bibliography	227
Index	229

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Preface

For about ten years between 1973 and 1986 the author was delivering a one-year topics course “Random Processes” at the Department of Mechanics and Mathematics of Moscow State University. This topics course was obligatory for third-fourth year undergraduate students (about 20 years of age) with major in probability theory and its applications. With great sympathy I remember my first students in this course: M. Safonov, A. Veretennikov, S. Anulova, and L. Mikhailovskaya. During these years the contents of the course gradually evolved, simplifying and shortening to the shape which has been presented in two 83 and 73 page long rotaprint lecture notes published by Moscow State University in 1986 and 1987. In 1990 I emigrated to the USA and in 1998 got the opportunity to present parts of the same course as a one-quarter topics course in probability theory for graduate students at the University of Minnesota. I thus had the opportunity to test the course in the USA as well as on several generations of students in Russia. What the reader finds below is a somewhat extended version of my lectures and the recitations which went along with the lectures in Russia.

The theory of random processes is an extremely vast branch of mathematics which cannot be covered even in ten one-year topics courses with minimal intersection of contents. Therefore, the intent of this book is to get the reader acquainted only with some parts of the theory. The choice of these parts was mainly defined by the duration of the course and the author’s taste and interests. However, there is no doubt that the ideas, facts, and techniques presented here will be useful if the reader decides to move on and study some other parts of the theory of random processes.

From the table of contents the reader can see that the main topics of the book are the Wiener process, stationary processes, infinitely divisible

processes, and Itô integral and stochastic equations. Chapters 1 and 3 are devoted to some techniques needed in other chapters. In Chapter 1 we discuss some general facts from probability theory and stochastic processes from the point of view of probability measures on Polish spaces. The results of this chapter help construct the Wiener process by using Donsker's invariance principle. They also play an important role in other issues, for instance, in statistics of random processes. In Chapter 3 we present basics of discrete time martingales, which then are used in one way or another in all subsequent chapters. Another common feature of all chapters excluding Chapter 1 is that we use stochastic integration with respect to random orthogonal measures. In particular, we use it for spectral representation of trajectories of stationary processes and for proving that Gaussian stationary processes with rational spectral densities are components of solutions to stochastic equations. In the case of infinitely divisible processes, stochastic integration allows us to obtain a representation of trajectories through jump measures. Apart from this and from the obvious connection between the Wiener process and Itô's calculus, all other chapters are independent and can be read in any order.

The book is designed as a textbook. Therefore it does not contain any new theoretical material but rather a new compilation of some known facts, methods and ways of presenting the material. A relative novelty in Chapter 2 is viewing the Itô stochastic integral as a particular case of the integral of nonrandom functions against random orthogonal measures. In Chapter 6 we give two proofs of Itô's formula: one is more or less traditional and the other is based on using stochastic intervals. There are about 128 exercises in the book. About 41 of them are used in the main text and are marked with an asterisk. The bibliography contains some references we use in the lectures and which can also be recommended as a source of additional reading on the subjects presented here, deeper results, and further references.

The author is sincerely grateful to Wonjae Chang, Kyeong-Hun Kim, and Kijung Lee, who read parts of the book and pointed out many errors, to Dan Stroock for his friendly criticism of the first draft, and to Naresh Jain for useful suggestions.

Nicolai Krylov
Minneapolis, January 2001

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Index

- A^c , 1
- $B_p^o(x)$, 13
- $\mathfrak{B}(X)$, 2, 6
- \mathfrak{B}^n , 15
- C , 16
- $D[0, T]$, 134
- $D[0, \infty)$, 134
- $E\{\xi|\mathcal{G}\}$, 71
- $E\{\xi|\zeta\}$, 71
- F_ξ , 4
- \mathcal{F}^P , 2
- \mathcal{F}_t^w , 52
- $\mathcal{F}_{s,t}^\xi$, 146
- \mathcal{F}_∞ , 83, 89
- \mathcal{F}_τ , 83
- H , 66
- H_0 , 169
- $L_p(\Pi, \mu)$, 39
- $L_2(0, 1)$, 47
- L_p -norm, 39
- l , 2
- \mathcal{M} , 195
- \mathfrak{M} , 211
- $N(a)$, 18
- $N(m, R)$, 22
- $P(A|\mathcal{G})$, 71
- $P\xi^{-1}$, 4
- \mathcal{P} , 61
- $R_{T,\varepsilon}$, 144
- \mathbb{R}_+ , 144
- $S(\Pi)$, 39
- \mathcal{S} , 179
- X^n , 15
- x_τ , 55
- \mathbb{Z}_n^d , 23
- $\beta(a, b)$, 87
- β_n , 134
- Δ_f , 10
- $\eta_t(f)$, 145
- λ -system, 45
- $\xi^{-1}(B)$, 3
- ξ_\pm , 4
- Π_0 , 39
- π -system, 45
- $\rho_t(b)$, 204
- $\Sigma(C)$, 16
- σ -field, 1
- σ -field generated by, 2, 4
- $\sigma(\mathcal{F})$, 2
- $\sigma(\xi)$, 4
- τ_a , 54
- $[\cdot]$, 5
- $\bigvee_n \mathcal{F}_n$, 89
- $\langle \xi \rangle$, 178
- $\mu_n \xrightarrow{w} \mu$, 10
- $(0, \gamma]$, 195
- $\|\cdot\|_p$, 39
- $\|\sigma\|$, 211
- adapted functions, 66
- almost surely, 3
- asymptotically normal sequences, 30
- Borel functions, 6
- Borel sets, 2, 6
- Borel σ -field, 2, 6
- cadlag functions, 134
- Cauchy process, 143
- centered Poisson measure, 156

- complete σ -field, 3
- completion of a σ -field, 2
- complex Wiener process, 189
- conditional expectation, 71
- continuous process, 16
- continuous-time random process, 14
- correlation function, 95
- covariance function, 22
- cylinder σ -field, 16

- defining sequence, 39
- distribution, 4
- Doob's decomposition, 79
- Doob's inequality for moments, 85
- Doob-Kolmogorov inequality, 84

- ergodic process, 125
- exchangeable sequence, 119
- expectation, 4
- exponential martingales, 205

- Feller property, 216
- filtration of σ -fields, 52, 81
- finite dimensional cylinder sets, 16
- finite-dimensional distribution, 15

- Gaussian process, 22, 104
- Gaussian vector, 21, 104
- generator of a process, 216

- independence, 52, 55
- independent processes, 55, 146
- infinitely divisible process, 137
- invariance principle, 32
- invariant event, 122
- Itô stochastic integral, 63, 171
- Itô's formula, 193

- jump measure, 144

- Khinchin's formula, 140

- Lebesgue σ -field, 3
- Lévy measure, 140
- Lévy's formula, 140

- Markov process, 220
- martingale, 78
- mean-square differentiability, 118
- mean-square integral, 101
- measurable space, 1
- modification of a process, 20
- multidimensional Wiener process, 186
- multiplicative decomposition, 79

- normal correlation theorem, 76
- normal vectors, 21

- number of upcrossings, 87

- Ornstein-Uhlenbeck process, 107

- Parseval's equality, 49
- path, 14
- Poisson process, 41, 42, 143
- Polish space, 6
- positive definite function, 95
- predictable functions, 61
- probability measure, 2
- probability space, 2
- processes bounded in probability, 132

- random field, 14
- random orthogonal measure, 40
- random process, 14
- random sequence, 14
- random spectral measure, 105
- random variable, 4
- reference measure, 40
- regular measure, 7
- relatively weakly compact family, 10
- reverse martingale, 80

- scalar product, 39
- Scheffé's theorem, 89
- second-order stationary process, 95
- self-similarity, 31, 50
- simple stopping time, 195
- spectral density, 98
- spectral measure, 98
- spectral representation, 105
- stable processes, 58
- standard random orthogonal measure, 108
- stationary process, 119
- step function, 39
- stochastic differential, 188
- stochastic integral, 44
- stochastic interval, 195
- stochastically continuous process, 132
- stopped sequences, 82
- stopping time, 54, 81
- submartingale, 78
- supermartingale, 78

- time homogeneous process, 137
- trajectory, 14

- Wald's distribution, 57
- Wald's identity, 177, 184
- weak convergence, 10
- white noise, 110
- Wiener measure, 30
- Wiener process, 52
- Wiener process relative to a filtration, 52

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(Continued from the front of this publication)

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