Several Complex Variables with Connections to Algebraic Geometry and Lie Groups

Joseph L. Taylor

Graduate Studies in Mathematics

Volume 46

American Mathematical Society

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http://dx.doi.org/10.1090/gsm/046

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Graduate Studies in Mathematics
Volume 46

American Mathematical Society
Providence, Rhode Island

2000 *Mathematics Subject Classification.* Primary 34–01, 14–01, 22–01, 43–01.

ABSTRACT. A graduate text with an integrated treatment of several complex variables and complex algebraic geometry, with applications to the structure theory and representation theory of Lie groups.

Library of Congress Cataloging-in-Publication Data

Taylor, Joseph L., 1941–
Several complex variables with connections to algebraic geometry and Lie groups / Joseph L. Taylor.
p. cm. — (Graduate studies in mathematics, ISSN 1065-7339 ; v. 46)
Includes bibliographical references and index.
ISBN 0-8218-3178-X (alk. paper)
1. Functions of several complex variables. 2. Geometry, Algebraic. I. Title. II. Series.

QA331.7.T39 2002
515′.94—dc21 2002018346

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Preface

This text evolved from notes I developed for use in a course on several complex variables at the University of Utah. The eclectic nature of the topics presented in the text reflects the interests and motivation of the graduate students who tended to enroll for this course. These students were almost all planning to specialize in either algebraic geometry or representation theory of semisimple Lie groups. The algebraic geometry students were primarily interested in several complex variables because of its connections with algebraic geometry, while the group representations students were primarily interested in applications of complex analysis – both algebraic and analytic – to group representations.

The course I designed to serve this mix of students involved a simultaneous development of basic complex algebraic geometry and basic several complex variables, which emphasized and capitalized on the similarities in technique of much of the foundational material in the two subjects. The course began with an exposition of the algebraic properties of the local rings of regular and holomorphic functions, first on $\mathbb{C}^n$ and then on varieties. This was followed by a development of abstract sheaf theory and sheaf cohomology and then by the introduction of coherent sheaves in both the algebraic and analytic settings. The fundamental vanishing theorems for both kinds of coherent sheaves were proved and then exploited. Typically the course ended with a proof and applications of Serre's GAGA theorems, which show the equivalence of the algebraic and analytic theories in the case of projective varieties. The notes for this course were corrected and refined, with the help of the students, each time the course was taught. This text is the result of that process.

There were instances where the course continued through the summer as a reading course for students in group representations. One summer, the objective was to prove the Borel-Weil-Bott theorem; another time, it was to explore a complex analysis approach to the study of representations of real semisimple Lie groups. Material from these summer courses was expanded and then included in the text as the final three chapters.

The material on several complex variables in the text owes a great debt to the text of Gunning and Rossi [**GR**], and the recent rewriting of that text by Gunning [**Gu**]. It was from Gunning and Rossi that I learned the subject, and the approach to the material that is used in Gunning and Rossi is also the approach used in this text. This means a thorough treatment of the local theory using the tools of commutative algebra, an extensive development of sheaf theory and the theory of coherent analytic sheaves, proofs of the main vanishing theorem for such sheaves (Cartan's Theorem B) in full generality, and a complete proof of the finite dimensionality of the cohomologies of coherent sheaves on compact varieties (the Cartan-Serre theorem). This does not mean that I have included treatments of all the topics covered in Gunning and Rossi. There is no discussion of pseudoconvexity, for example, or global embeddings, or the proper mapping theorem, or envelopes of holomorphy. I have included, however, a more extensive list of applications of the main results of the subject – particularly if one includes in this category Serre's GAGA theorems and the material on complex semisimple Lie groups and the proof of the Borel-Weil-Bott theorem.

Several complex variables is a very rich subject, which can be approached from a variety of points of view. The serious student of several complex variables should consult, not only Gunning's rewriting of Gunning and Rossi, but also the many excellent texts which approach the subject from other points of view. These include [**D**], [**Fi**], [**GRe**], [**GRe2**], [**Ho**], [**K**], and [**N**], to name just a few.

Interwoven with the material on several complex variables in this text is a simultaneous treatment of basic complex algebraic geometry. This includes the structure theory of local rings of regular functions and germs of varieties, dimension theory, the vanishing theorems for coherent and quasi-coherent algebraic sheaves, structure of regular maps between varieties, and the main theorems on the cohomology of coherent sheaves on projective spaces.

There are real advantages to this simultaneous development of algebraic and analytic geometry. Results in the two subjects often have essentially the same proofs; they both rely heavily on the same background material – commutative algebra for the local theory and homological algebra and sheaf theory for the global theory; and often a difficult proof in several complex variables can be motivated and clarified by an understanding of the often similar but technically simpler proof of the analogous result in algebraic geometry.

Several complex variables and complex algebraic geometry are not just similar; they are equivalent when done in the context of projective varieties. This is the content of Serre's GAGA theorems. We give complete proofs of these results in Chapter 13, after first studying the cohomology of coherent sheaves on projective spaces in Chapter 12.

The text could easily have ended with Chapter 13. This is where the course typically ends. The material in Chapters 14 through 16 is on quite a different subject – Lie groups and their representations – albeit one that involves the extensive use of several complex variables and algebraic geometry. Chapter 16 is devoted to a proof of the Borel-Weil-Bott theorem. This is the theorem which pinpoints the relationship between finite dimensional holomorphic representations of a complex semisimple Lie group G and the cohomologies of G-equivariant holomorphic line bundles on a projective variety, called the *flag variety*, constructed from G. Chapter 15 is a brief treatment of the subject of complex algebraic groups. This is included in order to provide proofs of some of the basic structure results for complex semisimple Lie groups that are needed in the formulation and proof of the Borel-Weil-Bott theorem. Chapter 14 is a survey of the background material needed if one is to understand Chapters 15 and 16. It includes material on topological groups and their representations, compact groups, Lie groups and Lie algebras, and finite dimensional representations of semisimple Lie algebras. These last three chapters are included primarily for the benefit of the student of Lie theory and group representations. This material illustrates that both several complex variables and complex algebraic geometry are essential tools in the modern study of group representations. The chapter on algebraic groups (Chapter 15) provides particularly compelling examples of the utility of algebraic geometry applied in the context of the structure theory of Lie groups. The proof of the Borel-Weil-Bott theorem in Chapter 16 involves applications of a wide range of material from several complex variables and algebraic geometry. In particular, it provides nice applications of the sheaf theory of Chapter 7, the Cartan-Serre theorem from Chapter 11, the material on projective varieties in Chapter 12, Serre's theorems in Chapter 13, and of course, the background material on algebraic groups and general Lie theory from Chapters 14 and 15.

I have tried to make the text as self-contained as possible. However, students who attempt to use it will need some background. This should include knowledge of the material from typical first year graduate courses in real and complex analysis, modern algebra, and topology. Also, students who wishes to confront the material in Chapters 14 through 16 will be helped greatly if they have had a basic introduction to Lie theory. Though the background material in Chapter 14 is reasonably self-contained, it is intended as a survey, and so some of the more technical proofs have been left out. For example, the basic theorems relating Lie algebras and Lie groups are stated without proof, as is the existence of compact real forms for complex semisimple groups and the classification of finite dimensional representations of semisimple Lie algebras.

Each chapter ends with an exercise set. Many exercises involve filling in details of proofs in the text or proving results that are needed elsewhere in the text, while others supplement the text by exploring examples or additional material. Cross-references in the text to exercises indicate both the chapter and the exercise number; that is, Exercise 2.5 refers to Exercise 5 of Chapter 2.

There are many individuals who contributed to the completion of this text. Edward Dunne, Editor for the AMS book program, noticed an early version of the course notes on my website and suggested that I consider turning them into a textbook. Without this suggestion and Ed's further advice and encouragement, the text would not exist. Several of my colleagues provided valuable ideas and suggestions. I received encouragement and much useful advice on issues in several complex variables from Hugo Rossi. Aaron Bertram, Herb Clemens, Dragan Miličić, Paul Roberts, and Angelo Vistoli gave me valuable advice on algebraic geometry and commutative algebra, making up, in part, for my lack of expertise in these areas. Henryk Hecht, Dragan Miličić, and Peter Trombi provided help on Lie theory and group representations. Without Dragan's help and advice, the chapters on Lie theory, algebraic groups, and the Borel-Weil-Bott theorem would not exist. The proof of the Borel-Weil-Bott theorem presented in Chapter 16 is due to Dragan, and he was the one who insisted that I approach structure theorems for semisimple Lie groups from the point of view of algebraic groups. The students who took the course the three times it was offered while the notes were being developed caught many errors and offered many useful suggestions. One of these students, Laura Smithies, after leaving Utah with a Ph.D. and taking a position at Kent State, volunteered to proofread the entire manuscript. I gratefully accepted this offer, and the result was numerous corrections and improvements. My sincere thanks goes out to all of these individuals and to my wife, Ulla, who showed great patience and understanding while this seemingly endless project was underway.

Joseph L. Taylor

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