

# Several Complex Variables with Connections to Algebraic Geometry and Lie Groups

**Joseph L. Taylor**

**Graduate Studies  
in Mathematics**

**Volume 46**



**American Mathematical Society**

# Selected Titles in This Series

<http://dx.doi.org/10.1090/gsm/046>

- 46 **Joseph L. Taylor**, Several complex variables with connections to algebraic geometry and Lie groups, 2002
- 45 **Inder K. Rana**, An introduction to measure and integration, second edition, 2002
- 44 **Jim Agler and John E. McCarthy**, Pick interpolation and Hilbert function spaces, 2002
- 43 **N. V. Krylov**, Introduction to the theory of random processes, 2002
- 42 **Jin Hong and Seok-Jin Kang**, Introduction to quantum groups and crystal bases, 2002
- 41 **Georgi V. Smirnov**, Introduction to the theory of differential inclusions, 2002
- 40 **Robert E. Greene and Steven G. Krantz**, Function theory of one complex variable, 2002
- 39 **Larry C. Grove**, Classical groups and geometric algebra, 2002
- 38 **Elton P. Hsu**, Stochastic analysis on manifolds, 2002
- 37 **Hershel M. Farkas and Irwin Kra**, Theta constants, Riemann surfaces and the modular group, 2001
- 36 **Martin Schechter**, Principles of functional analysis, second edition, 2002
- 35 **James F. Davis and Paul Kirk**, Lecture notes in algebraic topology, 2001
- 34 **Sigurdur Helgason**, Differential geometry, Lie groups, and symmetric spaces, 2001
- 33 **Dmitri Burago, Yuri Burago, and Sergei Ivanov**, A course in metric geometry, 2001
- 32 **Robert G. Bartle**, A modern theory of integration, 2001
- 31 **Ralf Korn and Elke Korn**, Option pricing and portfolio optimization: Modern methods of financial mathematics, 2001
- 30 **J. C. McConnell and J. C. Robson**, Noncommutative Noetherian rings, 2001
- 29 **Javier Duoandikoetxea**, Fourier analysis, 2001
- 28 **Liviu I. Nicolaescu**, Notes on Seiberg-Witten theory, 2000
- 27 **Thierry Aubin**, A course in differential geometry, 2001
- 26 **Rolf Berndt**, An introduction to symplectic geometry, 2001
- 25 **Thomas Friedrich**, Dirac operators in Riemannian geometry, 2000
- 24 **Helmut Koch**, Number theory: Algebraic numbers and functions, 2000
- 23 **Alberto Candel and Lawrence Conlon**, Foliations I, 2000
- 22 **Günter R. Krause and Thomas H. Lenagan**, Growth of algebras and Gelfand-Kirillov dimension, 2000
- 21 **John B. Conway**, A course in operator theory, 2000
- 20 **Robert E. Gompf and András I. Stipsicz**, 4-manifolds and Kirby calculus, 1999
- 19 **Lawrence C. Evans**, Partial differential equations, 1998
- 18 **Winfried Just and Martin Weese**, Discovering modern set theory. II: Set-theoretic tools for every mathematician, 1997
- 17 **Henryk Iwaniec**, Topics in classical automorphic forms, 1997
- 16 **Richard V. Kadison and John R. Ringrose**, Fundamentals of the theory of operator algebras. Volume II: Advanced theory, 1997
- 15 **Richard V. Kadison and John R. Ringrose**, Fundamentals of the theory of operator algebras. Volume I: Elementary theory, 1997
- 14 **Elliott H. Lieb and Michael Loss**, Analysis, 1997
- 13 **Paul C. Shields**, The ergodic theory of discrete sample paths, 1996
- 12 **N. V. Krylov**, Lectures on elliptic and parabolic equations in Hölder spaces, 1996
- 11 **Jacques Dixmier**, Enveloping algebras, 1996 Printing
- 10 **Barry Simon**, Representations of finite and compact groups, 1996
- 9 **Dino Lorenzini**, An invitation to arithmetic geometry, 1996
- 8 **Winfried Just and Martin Weese**, Discovering modern set theory. I: The basics, 1996

*(Continued in the back of this publication)*

*This page intentionally left blank*

Several Complex Variables  
with Connections  
to Algebraic Geometry  
and Lie Groups

*This page intentionally left blank*

# Several Complex Variables with Connections to Algebraic Geometry and Lie Groups

Joseph L. Taylor

Graduate Studies  
in Mathematics  
Volume 46



American Mathematical Society  
Providence, Rhode Island

## Editorial Board

Steven G. Krantz  
David Saltman (Chair)  
David Sattinger  
Ronald Stern

2000 *Mathematics Subject Classification*. Primary 34–01, 14–01, 22–01, 43–01.

ABSTRACT. A graduate text with an integrated treatment of several complex variables and complex algebraic geometry, with applications to the structure theory and representation theory of Lie groups.

---

### Library of Congress Cataloging-in-Publication Data

Taylor, Joseph L., 1941–

Several complex variables with connections to algebraic geometry and Lie groups / Joseph L. Taylor.

p. cm. — (Graduate studies in mathematics, ISSN 1065-7339 ; v. 46)

Includes bibliographical references and index.

ISBN 0-8218-3178-X (alk. paper)

1. Functions of several complex variables. 2. Geometry, Algebraic. I. Title. II. Series.

QA331.7.T39 2002  
515'.94—dc21

2002018346

---

**Copying and reprinting.** Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy a chapter for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication is permitted only under license from the American Mathematical Society. Requests for such permission should be addressed to the Acquisitions Department, American Mathematical Society, P. O. Box 6248, Providence, Rhode Island 02940-6248. Requests can also be made by e-mail to [reprint-permission@ams.org](mailto:reprint-permission@ams.org).

© 2002 by the American Mathematical Society. All rights reserved.

The American Mathematical Society retains all rights  
except those granted to the United States Government.  
Printed in the United States of America.

⊗ The paper used in this book is acid-free and falls within the guidelines  
established to ensure permanence and durability.

Visit the AMS home page at URL: <http://www.ams.org/>

10 9 8 7 6 5 4 3 2 1 07 06 05 04 03 02

---

# Contents

Preface	xiii
Chapter 1. Selected Problems in One Complex Variable	1
§1.1. Preliminaries	2
§1.2. A Simple Problem	2
§1.3. Partitions of Unity	4
§1.4. The Cauchy-Riemann Equations	7
§1.5. The Proof of Proposition 1.2.2	10
§1.6. The Mittag-Leffler and Weierstrass Theorems	12
§1.7. Conclusions and Comments	16
Exercises	18
Chapter 2. Holomorphic Functions of Several Variables	23
§2.1. Cauchy's Formula and Power Series Expansions	23
§2.2. Hartog's Theorem	26
§2.3. The Cauchy-Riemann Equations	29
§2.4. Convergence Theorems	29
§2.5. Domains of Holomorphy	31
Exercises	35
Chapter 3. Local Rings and Varieties	37
§3.1. Rings of Germs of Holomorphic Functions	38
§3.2. Hilbert's Basis Theorem	39



---

§3.3. The Weierstrass Theorems	40
§3.4. The Local Ring of Holomorphic Functions is Noetherian	44
§3.5. Varieties	45
§3.6. Irreducible Varieties	49
§3.7. Implicit and Inverse Mapping Theorems	50
§3.8. Holomorphic Functions on a Subvariety	55
Exercises	57
Chapter 4. The Nullstellensatz	61
§4.1. Reduction to the Case of Prime Ideals	61
§4.2. Survey of Results on Ring and Field Extensions	62
§4.3. Hilbert's Nullstellensatz	68
§4.4. Finite Branched Holomorphic Covers	72
§4.5. The Nullstellensatz	79
§4.6. Morphisms of Germs of Varieties	87
Exercises	92
Chapter 5. Dimension	95
§5.1. Topological Dimension	95
§5.2. Subvarieties of Codimension 1	97
§5.3. Krull Dimension	99
§5.4. Tangential Dimension	100
§5.5. Dimension and Regularity	103
§5.6. Dimension of Algebraic Varieties	104
§5.7. Algebraic vs. Holomorphic Dimension	108
Exercises	110
Chapter 6. Homological Algebra	113
§6.1. Abelian Categories	113
§6.2. Complexes	119
§6.3. Injective and Projective Resolutions	122
§6.4. Higher Derived Functors	126
§6.5. Ext	131
§6.6. The Category of Modules, Tor	133
§6.7. Hilbert's Syzygy Theorem	137
Exercises	142

---

Chapter 7. Sheaves and Sheaf Cohomology	145
§7.1. Sheaves	145
§7.2. Morphisms of Sheaves	150
§7.3. Operations on Sheaves	152
§7.4. Sheaf Cohomology	157
§7.5. Classes of Acyclic Sheaves	163
§7.6. Ringed Spaces	168
§7.7. De Rham Cohomology	172
§7.8. Čech Cohomology	174
§7.9. Line Bundles and Čech Cohomology	180
Exercises	182
Chapter 8. Coherent Algebraic Sheaves	185
§8.1. Abstract Varieties	186
§8.2. Localization	189
§8.3. Coherent and Quasi-coherent Algebraic Sheaves	194
§8.4. Theorems of Artin-Rees and Krull	197
§8.5. The Vanishing Theorem for Quasi-coherent Sheaves	199
§8.6. Cohomological Characterization of Affine Varieties	200
§8.7. Morphisms – Direct and Inverse Image	204
§8.8. An Open Mapping Theorem	207
Exercises	212
Chapter 9. Coherent Analytic Sheaves	215
§9.1. Coherence in the Analytic Case	215
§9.2. Oka’s Theorem	217
§9.3. Ideal Sheaves	221
§9.4. Coherent Sheaves on Varieties	225
§9.5. Morphisms between Coherent Sheaves	226
§9.6. Direct and Inverse Image	229
Exercises	234
Chapter 10. Stein Spaces	237
§10.1. Dolbeault Cohomology	237
§10.2. Chains of Syzygies	243
§10.3. Functional Analysis Preliminaries	245

---

§10.4. Cartan's Factorization Lemma	248
§10.5. Amalgamation of Syzygies	252
§10.6. Stein Spaces	257
Exercises	260
Chapter 11. Fréchet Sheaves – Cartan's Theorems	263
§11.1. Topological Vector Spaces	264
§11.2. The Topology of $\mathcal{H}(X)$	266
§11.3. Fréchet Sheaves	274
§11.4. Cartan's Theorems	277
§11.5. Applications of Cartan's Theorems	281
§11.6. Invertible Groups and Line Bundles	283
§11.7. Meromorphic Functions	284
§11.8. Holomorphic Functional Calculus	288
§11.9. Localization	298
§11.10. Coherent Sheaves on Compact Varieties	300
§11.11. Schwartz's Theorem	302
Exercises	309
Chapter 12. Projective Varieties	313
§12.1. Complex Projective Space	313
§12.2. Projective Space as an Algebraic and a Holomorphic Variety	314
§12.3. The Sheaves $\mathcal{O}(k)$ and $\mathcal{H}(k)$	317
§12.4. Applications of the Sheaves $\mathcal{O}(k)$	323
§12.5. Embeddings in Projective Space	325
Exercises	328
Chapter 13. Algebraic vs. Analytic – Serre's Theorems	331
§13.1. Faithfully Flat Ring Extensions	331
§13.2. Completion of Local Rings	334
§13.3. Local Rings of Algebraic vs. Holomorphic Functions	338
§13.4. The Algebraic to Holomorphic Functor	341
§13.5. Serre's Theorems	344
§13.6. Applications	351
Exercises	355

---

Chapter 14. Lie Groups and Their Representations	357
§14.1. Topological Groups	358
§14.2. Compact Topological Groups	363
§14.3. Lie Groups and Lie Algebras	376
§14.4. Lie Algebras	385
§14.5. Structure of Semisimple Lie Algebras	392
§14.6. Representations of $\mathfrak{sl}_2(\mathbb{C})$	400
§14.7. Representations of Semisimple Lie Algebras	404
§14.8. Compact Semisimple Groups	409
Exercises	416
Chapter 15. Algebraic Groups	419
§15.1. Algebraic Groups and Their Representations	419
§15.2. Quotients and Group Actions	423
§15.3. Existence of the Quotient	427
§15.4. Jordan Decomposition	430
§15.5. Tori	433
§15.6. Solvable Algebraic Groups	437
§15.7. Semisimple Groups and Borel Subgroups	442
§15.8. Complex Semisimple Lie Groups	451
Exercises	456
Chapter 16. The Borel-Weil-Bott Theorem	459
§16.1. Vector Bundles and Induced Representations	460
§16.2. Equivariant Line Bundles on the Flag Variety	464
§16.3. The Casimir Operator	469
§16.4. The Borel-Weil Theorem	474
§16.5. The Borel-Weil-Bott Theorem	478
§16.6. Consequences for Real Semisimple Lie Groups	483
§16.7. Infinite Dimensional Representations	484
Exercises	493
Bibliography	497
Index	501

*This page intentionally left blank*

---

# Preface

This text evolved from notes I developed for use in a course on several complex variables at the University of Utah. The eclectic nature of the topics presented in the text reflects the interests and motivation of the graduate students who tended to enroll for this course. These students were almost all planning to specialize in either algebraic geometry or representation theory of semisimple Lie groups. The algebraic geometry students were primarily interested in several complex variables because of its connections with algebraic geometry, while the group representations students were primarily interested in applications of complex analysis – both algebraic and analytic – to group representations.

The course I designed to serve this mix of students involved a simultaneous development of basic complex algebraic geometry and basic several complex variables, which emphasized and capitalized on the similarities in technique of much of the foundational material in the two subjects. The course began with an exposition of the algebraic properties of the local rings of regular and holomorphic functions, first on  $\mathbb{C}^n$  and then on varieties. This was followed by a development of abstract sheaf theory and sheaf cohomology and then by the introduction of coherent sheaves in both the algebraic and analytic settings. The fundamental vanishing theorems for both kinds of coherent sheaves were proved and then exploited. Typically the course ended with a proof and applications of Serre's GAGA theorems, which show the equivalence of the algebraic and analytic theories in the case of projective varieties. The notes for this course were corrected and refined, with the help of the students, each time the course was taught. This text is the result of that process.

There were instances where the course continued through the summer as a reading course for students in group representations. One summer, the objective was to prove the Borel-Weil-Bott theorem; another time, it was to explore a complex analysis approach to the study of representations of real semisimple Lie groups. Material from these summer courses was expanded and then included in the text as the final three chapters.

The material on several complex variables in the text owes a great debt to the text of Gunning and Rossi [**GR**], and the recent rewriting of that text by Gunning [**Gu**]. It was from Gunning and Rossi that I learned the subject, and the approach to the material that is used in Gunning and Rossi is also the approach used in this text. This means a thorough treatment of the local theory using the tools of commutative algebra, an extensive development of sheaf theory and the theory of coherent analytic sheaves, proofs of the main vanishing theorem for such sheaves (Cartan's Theorem B) in full generality, and a complete proof of the finite dimensionality of the cohomologies of coherent sheaves on compact varieties (the Cartan-Serre theorem). This does not mean that I have included treatments of all the topics covered in Gunning and Rossi. There is no discussion of pseudoconvexity, for example, or global embeddings, or the proper mapping theorem, or envelopes of holomorphy. I have included, however, a more extensive list of applications of the main results of the subject – particularly if one includes in this category Serre's GAGA theorems and the material on complex semisimple Lie groups and the proof of the Borel-Weil-Bott theorem.

Several complex variables is a very rich subject, which can be approached from a variety of points of view. The serious student of several complex variables should consult, not only Gunning's rewriting of Gunning and Rossi, but also the many excellent texts which approach the subject from other points of view. These include [**D**], [**Fi**], [**GR**], [**GR2**], [**Ho**], [**K**], and [**N**], to name just a few.

Interwoven with the material on several complex variables in this text is a simultaneous treatment of basic complex algebraic geometry. This includes the structure theory of local rings of regular functions and germs of varieties, dimension theory, the vanishing theorems for coherent and quasi-coherent algebraic sheaves, structure of regular maps between varieties, and the main theorems on the cohomology of coherent sheaves on projective spaces.

There are real advantages to this simultaneous development of algebraic and analytic geometry. Results in the two subjects often have essentially the same proofs; they both rely heavily on the same background material – commutative algebra for the local theory and homological algebra and sheaf theory for the global theory; and often a difficult proof in several complex variables can be motivated and clarified by an understanding of the often

similar but technically simpler proof of the analogous result in algebraic geometry.

Several complex variables and complex algebraic geometry are not just similar; they are equivalent when done in the context of projective varieties. This is the content of Serre's GAGA theorems. We give complete proofs of these results in Chapter 13, after first studying the cohomology of coherent sheaves on projective spaces in Chapter 12.

The text could easily have ended with Chapter 13. This is where the course typically ends. The material in Chapters 14 through 16 is on quite a different subject – Lie groups and their representations – albeit one that involves the extensive use of several complex variables and algebraic geometry. Chapter 16 is devoted to a proof of the Borel-Weil-Bott theorem. This is the theorem which pinpoints the relationship between finite dimensional holomorphic representations of a complex semisimple Lie group  $G$  and the cohomologies of  $G$ -equivariant holomorphic line bundles on a projective variety, called the *flag variety*, constructed from  $G$ . Chapter 15 is a brief treatment of the subject of complex algebraic groups. This is included in order to provide proofs of some of the basic structure results for complex semisimple Lie groups that are needed in the formulation and proof of the Borel-Weil-Bott theorem. Chapter 14 is a survey of the background material needed if one is to understand Chapters 15 and 16. It includes material on topological groups and their representations, compact groups, Lie groups and Lie algebras, and finite dimensional representations of semisimple Lie algebras. These last three chapters are included primarily for the benefit of the student of Lie theory and group representations. This material illustrates that both several complex variables and complex algebraic geometry are essential tools in the modern study of group representations. The chapter on algebraic groups (Chapter 15) provides particularly compelling examples of the utility of algebraic geometry applied in the context of the structure theory of Lie groups. The proof of the Borel-Weil-Bott theorem in Chapter 16 involves applications of a wide range of material from several complex variables and algebraic geometry. In particular, it provides nice applications of the sheaf theory of Chapter 7, the Cartan-Serre theorem from Chapter 11, the material on projective varieties in Chapter 12, Serre's theorems in Chapter 13, and of course, the background material on algebraic groups and general Lie theory from Chapters 14 and 15.

I have tried to make the text as self-contained as possible. However, students who attempt to use it will need some background. This should include knowledge of the material from typical first year graduate courses in real and complex analysis, modern algebra, and topology. Also, students who wishes to confront the material in Chapters 14 through 16 will be



helped greatly if they have had a basic introduction to Lie theory. Though the background material in Chapter 14 is reasonably self-contained, it is intended as a survey, and so some of the more technical proofs have been left out. For example, the basic theorems relating Lie algebras and Lie groups are stated without proof, as is the existence of compact real forms for complex semisimple groups and the classification of finite dimensional representations of semisimple Lie algebras.

Each chapter ends with an exercise set. Many exercises involve filling in details of proofs in the text or proving results that are needed elsewhere in the text, while others supplement the text by exploring examples or additional material. Cross-references in the text to exercises indicate both the chapter and the exercise number; that is, Exercise 2.5 refers to Exercise 5 of Chapter 2.

There are many individuals who contributed to the completion of this text. Edward Dunne, Editor for the AMS book program, noticed an early version of the course notes on my website and suggested that I consider turning them into a textbook. Without this suggestion and Ed's further advice and encouragement, the text would not exist. Several of my colleagues provided valuable ideas and suggestions. I received encouragement and much useful advice on issues in several complex variables from Hugo Rossi. Aaron Bertram, Herb Clemens, Dragan Miličić, Paul Roberts, and Angelo Vistoli gave me valuable advice on algebraic geometry and commutative algebra, making up, in part, for my lack of expertise in these areas. Henryk Hecht, Dragan Miličić, and Peter Trombi provided help on Lie theory and group representations. Without Dragan's help and advice, the chapters on Lie theory, algebraic groups, and the Borel-Weil-Bott theorem would not exist. The proof of the Borel-Weil-Bott theorem presented in Chapter 16 is due to Dragan, and he was the one who insisted that I approach structure theorems for semisimple Lie groups from the point of view of algebraic groups. The students who took the course the three times it was offered while the notes were being developed caught many errors and offered many useful suggestions. One of these students, Laura Smithies, after leaving Utah with a Ph.D. and taking a position at Kent State, volunteered to proofread the entire manuscript. I gratefully accepted this offer, and the result was numerous corrections and improvements. My sincere thanks goes out to all of these individuals and to my wife, Ulla, who showed great patience and understanding while this seemingly endless project was underway.

Joseph L. Taylor

*This page intentionally left blank*

---

# Bibliography

- [AM] Atiyah, M. F. and Macdonald, I. G., *Introduction to Commutative Algebra*, Addison-Wesley, 1969.
- [BB] Beilinson, A. and Bernstein, J., *Localisation de  $g$ -modules*, C. R. Acad. Sci., Paris, Sér. 1 **292** (1981), 15 – 18.
- [Bj] Björk, J.-E., *Analytic  $D$ -Modules and Applications*, Kluwer, 1993.
- [B] Bourbaki, N., *Groupes et Algèbres de Lie*, Masson, 1975.
- [Br] Bredon, G. E., *Sheaf Theory*, McGraw-Hill, 1967.
- [CE] Cartan, H. and Eilenberg, S., *Homological Algebra*, Princeton University Press, 1956.
- [CS] Cartan, H. and Serre, J. P., *Un théorème de finitude concernant les variétés analytiques compactes*, C. R. Acad. Sci., Paris **237** (1953), 128–130.
- [Ch] Chevalley, C., *Fondements de la géométrie algébriques*, Paris, 1958.
- [Ch2] Chevalley, C., *Une démonstration d'un théorème sur les groupes algébriques*, J. Mathématique Pure et Appliquées, **39(4)** (1960), 307–317.
- [E] Eisenbud, D., *Commutative Algebra with a View toward Algebraic Geometry*, Springer-Verlag, 1994.
- [D] D'Angelo, J. P., *Several Complex Variables and the Geometry of Real Hypersurfaces*, Springer-Verlag, 1994.
- [Fi] Field, M., *Several Complex Variables and Complex Manifolds I and II*, London Mathematical Society Lecture Note Series 65 and 66, Cambridge University Press, 1982.
- [Fo] Forster, O., *Lectures on Riemann Surfaces*, Springer-Verlag, 1981.
- [F] Frisch, J., *Points de platitude d'un morphisme d'espaces analytiques complexes*, Inventiones Math. **4** (1967), 118 – 138.
- [GM] Gelfand, S. and Manin, Yu., *Methods of Homological Algebra I: Introduction to Cohomology Theory and Derived Categories*, Springer-Verlag, 1991.
- [GP] Guillemin, V. and Pollack, A., *Differential Topology*, Prentice Hall, 1974.
- [Go] Godement, R., *Topologie Algébrique et Théorie des Faisceaux*, Hermann, 1958.

- [Gr] Grauert, H., *Analytische faserungen über holomorphvollständigen raumen*, Math. Annalen **135** (1958), 263–273.
- [GRe] Grauert, H. and Remmert, R., *Theory of Stein Spaces*, Springer-Verlag, 1979.
- [GRe2] Grauert, H. and Remmert, R., *Coherent Analytic Sheaves*, Springer-Verlag, 1984.
- [GPR] Grauert, H., Peternell, Th., and Remmert, R., *Several Complex Variables VII*, Encyclopaedia of Mathematical Sciences 74, Springer-Verlag, 1984.
- [Gro] Grothendieck, A., *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. **9** (1957), 119–221.
- [Gu] Gunning, R. C., *Introduction to Holomorphic Functions of Several Complex Variables*, Wadsworth and Brooks/Cole, 1990.
- [GR] Gunning, R. C. and Rossi, H., *Analytic Functions of Several Complex Variables*, Prentice-Hall, 1965.
- [H] Hartshorne, R., *Algebraic Geometry*, Springer-Verlag, 1977.
- [HM] Hecht, H., Miličić, D., Schmid, W., and Wolf, J., *Localization and standard modules for real semisimple Lie groups I: The duality theorem*, Inventiones Mathematicae **90** (1987), 297 – 332.
- [HT] Hecht, H. and Taylor, J. L., *Analytic localization of group representations*, Adv. in Math. **79** (1990), 139 – 212.
- [He] Helgason, S., *Differential Geometry and Symmetric Spaces*, Academic Press, 1962.
- [Ho] Hörmander, L., *Introduction to Complex Analysis in Several Variables*, North Holland, 1973.
- [Hum] Humphreys, J. E., *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, 1970.
- [Hum2] Humphreys, J. E., *Linear Algebraic Groups*, Springer-Verlag, 1981.
- [Hun] Hungerford, T. W., *Algebra*, Holt, Rinehart, and Winston, 1974.
- [KS] Kashiwara, M. and Schapira, P., *Sheaves on Manifolds*, Springer-Verlag, 1990.
- [KSc] Kashiwara, M. and Schmid, W., *Quasi-equivariant  $D$ -modules, equivariant derived category and representations of reductive Lie groups*, RIMS **980** (1994), 1 – 26.
- [K] Krantz S., *Function Theory of Several Complex Variables*, second edition, AMS, 1992.
- [Mat] Matsumura, H., *Commutative Algebra*, second edition, Benjamin/Cummings, 1980.
- [M] Miličić, D., *A proof of the Borel–Weil theorem*, unpublished notes.
- [Mac] Mac Lane, S., *Homology*, Springer-Verlag, 1967.
- [Mit] Mitchell, B., *Theory of Categories*, Pure App. Math. 17, Academic Press, 1965.
- [MZ] Montgomery, D. and Zippin, L., *Topological Transformation Groups*, Tracts in Pure and Applied Mathematics 1, Interscience, 1955.
- [Mum] Mumford, D., *The Red Book of Varieties and Schemes*, Lecture Notes in Math. 1358, Springer-Verlag, 1988.
- [Nai] Naimark, M. A., *Normed Algebras*, Wolters-Noordhoff, 1972.
- [N] Narasimhan, R., *Introduction to the Theory of Analytic Spaces*, Lecture Notes in Mathematics 25, Springer-Verlag, 1966.
- [Ro] Rotman, J., *Notes on Homological Algebra*, Reinhold Math. Studies 26, Van Nostrand, 1970.
- [R] Rudin, W., *Real and Complex Analysis*, McGraw-Hill, 1987.

- [R2] Rudin, W., *Functional Analysis*, McGraw - Hill, 1973.
- [R3] Rudin, W., *Fourier Analysis on Groups*, Interscience, 1962.
- [Sc] Schaefer, H. H., *Topological Vector Spaces*, Macmillan, 1966.
- [Sch] Schmid, W., *Boundary value problems for group invariant differential equations*, in Élie Cartan et les mathématiques d'Aujourd'hui, Astérisque numéro hors séries **6** (1985), 311–322.
- [S] Serre, J. P., *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier **6** (1956), 1–42.
- [S2] Serre, J. P., *Faisceaux algébriques cohérents*, Ann. of Math. **61** (1955), 197–278.
- [Siu] Siu, Y.-T., *Noetherianess of rings of holomorphic functions on Stein compact sets*, Proc. AMS **21** (1969), 483 – 489.
- [Sm] Smithies, L., *Equivariant analytic localization of group representations*, Memoirs of AMS **728** **153**.
- [T] Taylor, J. L., *A general framework for a multi-operator functional calculus*, Adv. in Math **9** (**2**) (1972), 183 – 252.
- [T2] Taylor, J. L., *Topological invariants of the maximal ideal space of a Banach algebra*, Adv. in Math **19** (1976), 149 – 206.
- [V] Varadarajan, V. S., *Lie Groups, Lie Algebras, and Their Representations*, Springer - Verlag, 1984.
- [V2] Varadarajan, V. S., *An Introduction to Harmonic Analysis on Semi-simple Lie Groups*, Cambridge Studies in Advanced Mathematics 16, Cambridge University Press, 1989.
- [Vo] Vogan, D. A., *Representations of Real Reductive Lie Groups*, Progress in Mathematics 15, Birkhäuser, 1981.
- [Wa] Warner, F. W., *Foundations of Differentiable Manifolds and Lie Groups*, Scott, Foresman, and Co., 1971.
- [W] Wells, R., *Differential Analysis on Complex Manifolds*, Springer-Verlag, 1980.
- [ZS] Zariski, O and Samuel, P., *Commutative Algebra*, Van Nostrand, 1960.

*This page intentionally left blank*

---

# Index

- abelian category, 116, 117
- abelian subcategory, 118
- abelian variety, 421
- acyclic
  - objects for a functor, 130
  - sheaves, 159
- additive category, 116
- additive functor, 116
- adjacent Weyl chambers, 479
- adjoint of an operator, 360
- adjoint representation, 384
- adjunction morphisms, 155
- admissible representation, 485
- affine variety, 186
- algebraic
  - element, 62
  - field extension, 62
  - group, 376, 419
  - group action, 424
  - prevariety, 186
  - sheaf, 194
  - subvariety, 45
  - variety, 187
- algebraically induced analytic sheaves, 347
- aligned pair of boxes, 249
- ample sheaf, 326
- analytic modules of finite type, 487
- analytic polyhedron, 261
- analytic sheaf, 216
- analytic vector, 486
- Arens-Royden theorem, 296
- Artin-Rees theorem, 197
  
- balanced set, 264, 304
- Banach algebra, 245, 288
- bidegree of a differential form, 238
  
- biholomorphic
  - equivalence, 51
  - mapping, 50, 55
- biholomorphically equivalent, 56
- biregular
  - mapping, 55
- Borel subalgebra, 399, 443
- Borel subgroup, 443
- Borel-Weil theorem, 477
- Borel-Weil-Bott theorem, 480
- bounded
  - complex, 119
  - subset of  $\mathcal{H}(U)$ , 30
- bounded subset, 264
- box in  $\mathbb{C}^n$ , 249
- branching order, 73
  
- Cartan subalgebra, 395
- Cartan's factorization lemma, 251
- Cartan's first criterion, 391
- Cartan's second criterion, 391
- Cartan's Theorem A, 278
- Cartan's Theorem B, 279
- Cartan-Serre theorem, 302
- Cartesian product
  - of prevarieties, 187
- Casimir operator, 469
- category, 114
  - abelian, 117
  - additive, 116
  - homotopy, 120
  - of functors, 115
  - of modules, 133
  - of morphisms, 115
- Cauchy integral formula
  - generalized, 8

- Cauchy's inequalities, 25, 30  
 Cauchy's integral formula, 24  
 Cauchy-Riemann equations, 7, 29  
 Čech cohomology, 176, 179  
 Čech complex  
   global, 176  
   limit, 179  
   of sheaves, 176  
 centralizer, 440, 442  
 chain of syzygies, 142  
 character, 362  
   of a representation, 375  
   of an algebraic group, 429  
   unitary, 362  
 characteristic polynomial, 67  
 Chern class, 181, 284  
 closed embedding, 326  
 closed graph theorem, 265  
 closure of modules theorem, 274  
 coherent  
   algebraic sheaf, 194  
   analytic sheaf, 216  
   sheaf of algebras, 213, 356  
 cohomology  
   Čech, 176, 179  
   of a complex, 119  
   sheaf, 159  
 coimage of a morphism, 117  
 cokernel of a morphism, 117  
 complete algebraic variety, 327  
 completion of a local ring, 334  
 complex  
   exact, 119  
   of maps, 12  
   of morphisms, 119  
 complex homomorphism, 290  
 complex orthogonal group, 421  
 constant presheaf, 147  
 constant sheaf, 150  
 continuous linear functional, 264  
 contractible space, 183  
 convex separation theorem, 265  
 convolution product, 363  
 cotangent space, 173  
 Cousin data, 13  
 Cousin problem, 12  
  
 d-group, 434  
 de Rham  
   cohomology, 174  
   complex, 174  
 dense regular subcover, 72  
 depth of a prime ideal, 110  
 derivation  
   of a Lie algebra, 393  
 derived functors, 127  
 diagonal group, 421  
 differential form, 7, 173  
 differential forms, 7  
 differentials of a complex, 119  
 dimension  
   Krull, 99  
   of a holomorphic variety, 96  
   of a submanifold, 51  
   of an algebraic variety, 104, 189  
   pure, 97, 104  
   tangential, 100  
 direct image, 153  
   with proper supports, 157  
 direct image functor, 154  
 discriminant, 64, 82  
 divisible group, 134  
 divisor, 285  
 Dolbeault cohomology, 239  
 Dolbeault complex, 239  
 Dolbeault's lemma, 241  
 domain of convergence, 36  
 domain of holomorphy, 32–35  
 dual basis, 469  
  
 elementary symmetric functions, 41, 43  
 embedding, 326  
   of algebraic varieties, 204  
 Engel's theorem, 386  
 enough injectives, 124  
 enough projectives, 125  
 enveloping algebra, 469  
 epimorphism, 114  
 equivalence of categories, 115  
 equivariant  
   vector bundle, 460  
 equivariant vector bundle, 460  
 exact  
   complex, 119  
   functor, 118, 122  
   sequence, 11, 118  
 exceptional coordinate, 249  
 exponential map, 382  
 Ext, 131  
 exterior differentiation, 239  
  
 faithfully flat ring extension, 332  
 family of supports, 158  
 fine sheaf, 163  
 finite  
   algebra over a ring, 63, 79, 80  
   extension, 63, 80, 81, 89  
   morphism of algebraic varieties, 111  
 finite branched holomorphic cover, 72, 74, 83, 89  
 finite extension, 83  
 finite holomorphic covering map, 72  
 finite holomorphic map, 230  
 finite morphism, 87, 206



- finite vanishing order, 40, 80
- flabby sheaf, 163
- flag, 446
  - full, 446
  - variety, 446
- flat
  - module, 136
  - morphism, 205
- Forster's theorem, 297
- Fourier inversion theorem, 362
- Fourier transform, 362
- Fréchet sheaf, 274
- Fréchet space, 30
- free module, 133
- full subcategory, 114
- functor, 115
  - $\delta$ -functor, 129
  - additive, 116
  - direct image, 154
  - exact, 118, 122
  - inverse image, 153
  - left exact, 123
  - restriction, 154
- fundamental group, 381
- Galois
  - group, 63
- Galois extension, 63
- Gelfand transform, 291
- general linear group, 420
- generalized eigenspace, 389
- geometric fiber, 347
- germ
  - of a function, 38
  - of a holomorphic variety, 46
  - of an algebraic variety, 46
  - of an element of a presheaf, 148
- Going up theorem, 70
- graded ring, 197
- graph of a linear map, 265
- Grothendieck universe, 113, 127
- group action
  - holomorphic, 378
  - on a space, 377
- Haar measure, 359
- Hahn-Banach theorem, 265
- Harish-Chandra module, 486
- Hartog's
  - lemma, 27
  - theorem, 28
- height of a prime ideal, 110
- Hilbert space, 360
- Hilbert's basis theorem, 39
- Hilbert's Nullstellensatz, 68, 70
- Hilbert's syzygy theorem, 142
- holomorphic
  - Banach space valued function, 20
  - extension, 32
  - function, 2, 25
  - function on a subvariety, 55
  - functional calculus, 288, 293
  - mapping, 35, 55
  - $p$ -form, 240
  - submanifold, 50
  - subvariety, 45
  - variety, 170
- holomorphically convex, 33
  - compact set, 257
  - hull, 33, 257
  - open set, 257
  - variety, 257
- homogeneous ideal, 328
- homotopic
  - maps, 183
  - morphisms of complexes, 120
- homotopy category, 120
- hyperplane in projective space, 347
- ideal
  - of a Lie algebra, 380
  - of a variety, 47
- ideal sheaf, 200
  - of a subvariety, 221
- image
  - of a morphism, 117
- immersion, 382
- implicit function theorem, 51
- implicit mapping theorem, 51
- inclusion morphisms, 116
- induced bundle, 461
- induction, 460
- injective
  - object, 123
  - resolution, 124
- inner product, 360
- integral
  - element, 63
  - extension, 63
- integral subgroup, 382
- integrally closed, 64
- interpolation theorem, 19
- intertwining operator, 359
- invariant subspace, 358
- inverse image, 153
- inverse image functor, 153
  - algebraic, 205
  - analytic, 229
- inverse mapping theorem, 52, 103
- inverse of a morphism, 114
- invertible sheaves, 181
- involution, 360
- irreducible
  - germ of a variety, 49

- subvariety, 49
  - variety, 189
- irreducible components, 189
- irreducible representation, 358
- isomorphism in a category, 114
- isotropy group, 424
- Jacobi identity, 379
- Jacobian, 51
- Jensen's inequality, 26, 28
- joint spectrum, 289
- Jordan decomposition
  - abstract, 394
  - abstract multiplicative, 432
  - multiplicative, 431
  - of a matrix, 388
- Jordan-Chevalley decomposition lemma, 388
- kernel of a morphism, 117
- Killing form, 388
- Krull dimension, 99
- Krull's theorem, 198
- left exact functor, 123
- length of a Weyl group element, 480
- Leray cover, 177
- Lie algebra, 379
  - compact, 409
  - nilpotent, 385
  - of a Lie group, 380
  - semisimple, 385
  - solvable, 385
- Lie correspondence, 380
- Lie group, 358, 376
  - algebraic, 419
  - complex, 358, 376
  - real, 376
- Lie subalgebra, 380
- Lie subgroup, 377
- Lie's theorem, 387
- line bundle, 180
- local parameterization theorem, 85
- local ring, 39
- localization
  - algebraic, 191
  - analytic, 298
- locally closed set, 162
- locally convex topological vector space, 264
- locally finitely generated analytic sheaf, 216
- locus of an ideal, 47
- Mackey-Arens theorem, 307
- manifold
  - $C^\infty$ , 170
  - complex analytic, 170
  - topological, 170
- matrix coefficient, 371, 454
- maximal ideal space, 291
- maximum modulus theorem, 35
- meromorphic function, 13, 19, 285
- Meyer-Vietoris sequence, 168
- minimal globalization, 487
- minimal polynomial, 62, 81
- Mittag-Leffler theorem, 13
- module
  - free, 133
  - projective, 133
- monomorphism, 114
- Montel space, 30, 264
- morphism
  - finite, 206
  - of complexes, 119
  - of functors, 115
  - of germs of varieties, 87
  - of Lie algebras, 380
  - of Lie groups, 376
  - of morphisms, 115
  - of presheaves, 146
  - of ringed spaces, 168
  - of sheaves, 150
  - of vector bundles, 171
- morphisms of a category, 114
- Nakayama's lemma, 56
- neatly embedded, 102
- nilpotent
  - Lie algebra, 385
- nilpotent element, 386
- Noether normalization theorem
  - generalized, 93
- Noether property, 298
- Noether's normalization theorem, 68
- Noetherian ring, 39
- non-singular
  - algebraic variety, 107
  - germ of a holomorphic variety, 51
  - point of a holomorphic subvariety, 51
- norm for a finite ring extension, 67, 98
- normal
  - germ of a variety, 91
  - point of a variety, 91
- normal domain, 64
- Nullstellensatz, 61
- objects of a category, 114
- Oka's theorem, 219
- Oka-Weil subdomain, 258
- open mapping theorem, 265
- operator
  - hermitian, 360
  - self-adjoint, 360
  - unitary, 360
- orbit of a group action, 424
- orthogonal

- complement, 360
- vectors, 360
- orthogonal projection, 360
- Osgood's lemma, 24, 28, 30
- partition of unity, 5
- Peter-Weyl theorem, 375
- Picard group
  - of a ringed space, 181, 283
  - of an algebra, 297
- Plancherel theorem, 362
- Poincaré lemma, 174
- pole set, 286
- polydisc, 24
- polynomial polyhedron, 294
- Pontryagin duality theorem, 362
- power sum functions, 41
- power sums, 43
- presheaf, 146
  - constant, 147
  - skyscraper, 147
- principle part
  - of a meromorphic function, 13
- principle series representations, 489
- probability measure, 364
- projection morphisms, 116
- projective
  - module, 133
  - object, 123
  - resolution, 125
- projective morphism, 330
- projective space, 313
- projective variety, 316
- proper map, 58, 72, 156
- pure dimension, 97, 104
- pure order
  - of a cover, 73
- quasi-coherent sheaf, 194
- quasi-inverse, 115
- quasi-isomorphism of complexes, 125
- quasi-projective variety, 316
- radical
  - of a Lie algebra, 392, 442
  - of a Lie group, 442
  - of an ideal, 49
- real form
  - of a Lie algebra, 413
  - of a Lie group, 401, 413
- reducible
  - germ of a variety, 49
  - subvariety, 49
- regular
  - element of a torus, 436
  - function, 38
  - function on a subvariety, 55
  - germ of a holomorphic variety, 51
  - germ of an algebraic variety, 107
  - local ring, 141
  - mapping, 55
  - point of a holomorphic subvariety, 51
  - point of an algebraic variety, 107, 189
  - $V$ -valued function, 421
  - weight, 476
- regular Borel measure, 359
- regular locus, 95
- regular representation, 362
- regular system of parameters, 141
- removable singularity theorem, 41
- representation
  - adjoint, 384
  - holomorphic, 384
  - irreducible, 358
  - left regular, 371
  - of a Lie algebra, 384
  - of a topological group, 358
  - of an algebraic group, 422
  - right regular, 371
- resolution
  - injective, 124
  - projective, 125
- resolvent set, 289
- restriction
  - functor, 154
  - map, 146
  - of a sheaf, 154
- Riemann sphere, 188
- ringed space, 168
- root space decomposition, 395
- root spaces, 395
- roots
  - of a Cartan subalgebra, 395
  - positive system of, 398
- Rückert's Nullstellensatz, 68, 85
- saturated family of bounded sets, 304
- Schur's lemma, 360, 369
- Schwartz inequality, 360
- Schwarz's lemma, 35
- section
  - of a presheaf, 149
  - of a vector bundle, 171
- seminorm, 29, 264
- semisimple
  - algebraic group, 442
  - element, 388
  - group element, 432
  - Lie algebra, 385
  - Lie group, 409
  - linear operator, 430
- separated quotient, 265
- sheaf, 147
  - acyclic, 159

- algebraic, 194
- ample, 326
- analytic, 216
- coherent analytic, 216
- cohomology, 159
- constant, 150
- fine, 163
- flabby, 163
- Fréchet, 274
- invertible, 181
- morphism, 150
- of alternating  $p$ -cochains, 175
- of commutative rings, 152
- of divisors, 285
- of homomorphisms, 157
- of meromorphic functions, 285
- of modules, 152
- of  $p$ -cochains, 174
- of sections of a presheaf, 149
- soft, 163
- very ample, 326
- Shilov's Idempotent theorem, 295
- short exact sequence, 120
- simply connected, 381
- singular
  - germ of a holomorphic variety, 51
  - germ of an algebraic variety, 107
  - locus, 95
  - locus of an algebraic variety, 107
  - point of a holomorphic subvariety, 51
  - point of an algebraic variety, 107, 189
  - weight, 478
- skyscraper presheaf, 147
- smooth algebraic variety, 107, 189
- soft sheaf, 163
- solvable
  - Lie algebra, 385
- solvable algebraic group, 437
- special linear group, 421
- spectra radius, 289
- spectral theorem, 361
- spectrum, 289
- split
  - epimorphism, 124
  - monomorphism, 124
  - short exact sequence, 124
- splitting field, 63
- stabilizer, 424, 427
- stalk of a presheaf, 148
- Stein compact set, 282
- Stein manifold, 285
- Stein space, 257
- structure sheaf, 168
- subcategory, 114
  - full, 114
- subvariety
  - algebraic, 45, 188
  - holomorphic, 45
  - support of a section, 158
  - symplectic group, 421
  - system of simple roots, 418
  - syzygies
    - chain of, 142
- tangent
  - bundle, 173
  - space, 100
  - vector, 100
- tangential dimension, 100
- tensor product
  - of modules, 135
  - of sheaves, 157
- Theorem of the primitive element, 62
- thin subset, 41, 86
- topological dimension, 95
- topological group, 245, 358
- topological vector space, 264
- Tor, 136
- toral subalgebra, 395
- total degree of a differential form, 238
- totally bounded set, 305
- triangular group, 421
- trivial bundle, 171
- unipotent
  - algebraic group, 438
  - group element, 432
  - linear operator, 430
- unitary group of Hilbert space, 360
- universal covering group, 381
- Vandermonde determinant, 65
- vanishing order, 40, 51
- variety
  - affine, 186
  - algebraic, 187
  - complete, 327
  - holomorphic, 170
  - irreducible, 189
- vector bundle, 171
- vector field, 173
  - holomorphic, 380
  - right invariant, 380
- very ample, 326
- Vietoris-Begle theorem, 161
- wall between Weyl chambers, 479
- Weierstrass division theorem, 43
- Weierstrass polynomial, 42
- Weierstrass preparation theorem, 42
- Weierstrass theorem, 14
- weight, 402, 404, 429, 436
  - dominant, 409
  - highest, 402

integral, 409  
  space, 402, 404, 429  
  vector, 429  
Weyl chamber, 406  
  positive, 406  
Weyl group, 407  
Weyl system, 413, 415  
  
Zariski topology, 38

*This page intentionally left blank*

# Selected Titles in This Series

*(Continued from the front of this publication)*

- 7 **Gerald J. Janusz**, Algebraic number fields, second edition, 1996
- 6 **Jens Carsten Jantzen**, Lectures on quantum groups, 1996
- 5 **Rick Miranda**, Algebraic curves and Riemann surfaces, 1995
- 4 **Russell A. Gordon**, The integrals of Lebesgue, Denjoy, Perron, and Henstock, 1994
- 3 **William W. Adams and Philippe Loustau**, An introduction to Gröbner bases, 1994
- 2 **Jack Graver, Brigitte Servatius, and Herman Servatius**, Combinatorial rigidity, 1993
- 1 **Ethan Akin**, The general topology of dynamical systems, 1993

ISBN 0-8218-3178-X



9 780821 831786

GSM/46

AMS *on the Web*  
[www.ams.org](http://www.ams.org)