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ABSTRACT. This textbook develops the theory of secondary cohomology operations for singular cohomology theory and makes applications in the form of explicit calculations. The treatment is intended for graduate students with a knowledge of basic algebraic topology including exposure to the Steenrod operations. The subject is developed in terms of elementary constructions from general homotopy theory. Among the applications, there are proofs of the Hopf invariant one theorems for all primes. The final chapter treats the theory of Massey-Peterson fibrations in order to enlarge the scope of the basic theory through universal examples. This chapter also includes twisted operations and Cartan formulas.

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In memory of Alexander Zabrodsky (1936–1986).

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Preface

Secondary cohomology operations are one of the tools available which bear on questions left unresolved by primary operations. This book develops this specialized topic in terms of elementary concepts from general homotopy theory. The special circumstances of their applications mean that secondary operations are often found embedded in detailed computations and other technicalities. It has been known for some time that the subject can be set up in elementary terms. Perhaps more recent is the understanding that there are systematic strategies for making calculations which can also be developed in the same elementary framework. For the author, that understanding emerged in joint work with Alex Zabrodsky.

The first six chapters of this book develop the subject along the lines alluded to above. This work takes us through a proof of the Hopf invariant one theorem of J. F. Adams for the prime 2 and Liulevicius, Shimada and Yamanoshita for odd primes. Our proofs are in the spirit of those works but do not rely on calculations of the cohomology of universal examples. Moreover, by applying the elegant method employed by Shimada and Yamanoshita on the Steenrod algebra, the relation leading to the factorization of the appropriate Steenrod operation by secondary operations is worked out without working through the cohomology of the Steenrod algebra.

Besides the Hopf invariant one theorem, many other results about classical secondary operations are presented in the first six chapters. Notable among these is Browder's evaluation of higher order Bocksteins on p-th powers.

Our approach to the subject is through the idea of secondary compositions. This is an old idea, having great success in the hands of Barratt and Toda, among others, in the study of the homotopy groups of spheres. Many people have realized that secondary compositions supply a description of secondary cohomology operations and Spanier published an account of the basic theory. Nevertheless, it usually transpires that to make calculations, one must rely on ad hoc information in many cases or on advanced methods developed for the Adams spectral sequence.

In our work on finite H-spaces, Zabrodsky and I were confronted with the evaluation of a certain p-th order operation which was inaccessible by any method we knew at that time. We were able to resolve our problem by using the Milnor filtration where a space is regarded as the classifying space of its loop space. We realized that this method gave an alternative path through most of the literature where secondary operations were calculated and Alex gave a series of lectures in this vein for a workshop held in Barcelona. The idea that a textbook devoted to a similar treatment might be useful comes from the fact that the method still finds uses and the belief that the subject has both elegance and scope.

The table of contents indicates what may be found. Here I want to make some marginal comments on the material. I expect that readers of this book are familiar with the Steenrod algebra and its uses. For many people this means knowing the basic properties through the Adem relations and Milnor's structure theorem for the Hopf algebra. The first chapter interweaves a geometric discussion of primary operations with a summary discussion of features of the Steenrod operations. Also present is one of the systematic strategies for calculations. It is an argument first given by Adem to study compositions. I call it the Adem argument to indicate its general nature.

On the algebraic side, chapter one contains a new proof of a theorem due to A. Negishi concerning certain left multiplications in the Steenrod algebra. This result is used at the prime 2 to give the same argument that Shimada and Yamanoshita give at odd primes for the relation factoring certain Steenrod operations through secondary operations. Naturally the level of detail here exceeds that of a summary discussion. I have also included a largely unnecessary discussion of the ideal in the Steenrod algebra consisting of operations annihilating classes of a fixed dimension. This material is included because it simplifies some of my early work on the subject.

The overall purpose of the first chapter is to have the Steenrod algebra and the cohomology of Eilenberg-Mac Lane spaces ready for use in subsequent work. I have not tried to develop these topics, even in sketch form.

Our treatment of secondary operations deviates from the approach found in most of the literature. I will to try to delineate the differences. Typically, a secondary operation is produced in the cohomology of a universal example. This approach, where an element is picked out of a module, does not come with means for evaluation in specific cases. Our approach is to represent the cohomology class by a map known as a colifting defined in homotopy theoretic terms from the same data defining the operation. Then an evaluation of the operation is represented by a secondary composition. This simple geometric description entails a general formula for making calculations. In the literature, it is known as the Peterson-Stein formula or compatibility with connecting homomorphisms in Cartan's treatment. In elementary terms, the method is simply an adjoint relationship appearing in diagrams which can be recognized in many calculations. In the work with Zabrodsky mentioned above, we found exactly these patterns presented by the Milnor filtration. For us, the key feature provided by this filtration is an analysis of essential maps which become null-homotopic upon looping. Our direct hold on this phenomenon is another place where our treatment differs from most of the literature. There, the phenomenon is encoded in terms of splittings of universal examples as spaces, but not as H-spaces.

Chapter two is a bridge to the point of view dominating our development. Chapter three presents the basic geometric theory including our version of the Peterson-Stein formula and our use of the Milnor filtration. I have tried to separate those elements which appear to be general from those particular to secondary operations. Chapter four develops the basic theory of secondary operations. Except for the language, there is no difference between the results of our development and the traditional ones found in Adams' paper, the Cartan seminar, or the book by Mosher and Tangora. In chapter four we also indicate how our theory applies to operations of order higher than secondary operations. However, we do not develop this aspect in a systematic way. It is my opinion that technical matters get in the way of understanding unless one already has a good hold on the secondary situation. Moreover, I am unaware of Milnor filtration type information for the universal examples that serve higher order operations except for the case of higher order Bocksteins.

Chapter five presents several examples where the Milnor filtration comes into the story. All these calculations appear as direct applications of the basic strategies, strategies arising from recognition of a common pattern described in chapter two.

In particular, we have the evaluation of Adams and Liulevicius-Shimada-Yamanoshita operations in the cohomology of complex projective space. Moreover, the algebraic part of the decomposition formula is produced in this chapter. Thus the means to settle Hopf invariant one are present but the denouement is delayed until the next chapter.

Readers wishing to follow a connected account of the Hopf invariant one theorems can do so by leaving chapter three after Prop. 3.5.3 ((a) if

$p = 2$ is preferred) omitting the material in chapter four after subsection 4.2.8 (exercises 4.2.3–5 may be omitted) and going to section 5.3 (5.4 for odd primes).

Chapter six contains the Hopf invariant one results. The first part includes classical background material following lectures of John Moore. The second part assembles previous work to finish the proofs for the cases left open by the classical work.

As may be inferred, we do not base our work on the cohomology of universal examples, but it would be perverse to ignore this topic. Chapter seven is devoted in part to the work of Massey and Peterson which provides the most comprehensive hold on the cohomology structure of the spaces arising in the classical theory. The geometric work of earlier chapters takes its place in this theory, in particular, in the discussion of the Hopf algebra structure and the principal action.

Chapter seven also includes a discussion of twisted operations and Cartan formulas for secondary operations. I know that unrestrained glee is inappropriate to a sober preface, but let me say that I was pleased to find that the material of chapters three and four could sustain the discussion of these topics.

It is my belief that the ideas developed in this text can continue to be of use in homotopy theory. The book includes many examples and exercises with the intention that the reader will work through these as the principal means to understand the subject. Some of these examples, especially in chapter seven, are important for applications, notably in obstruction theory. However, I have only supplied references and have not tried to sketch the applications themselves. In fact, an excellent exercise is to look at the references and rework the relevant parts. Who knows, maybe that will lead to further understanding of these classical topics.

I mentioned earlier that the point of view for this book grew out of work with Alex Zabrodsky. I like to think that Alex would have enjoyed sharing authorship of this book. I would like to think that his influence here is undiminished, but I know that is not true. This book in dedicated in memoriam to Alex as thanks for the pleasure of working with him and for his profound contributions to homotopy theory.

Acknowledgement

It is a pleasure to thank people who have helped me with this book. Foremost is Joan Robinson, who prepared this book from a handwritten manuscript. Both John Moore and Haynes Miller supplied me with detailed comments on the early sections. Joe Roitberg attended a graduate course I offered in 1990 and Fred Cohen and Moore did the same in 1994. Their presence allowed me to try out various ways to present the material and they were not reluctant to tell me when something did not work. The graduate students in those courses also deserve my thanks for their attendance. Sam Gitler has listened patiently to my explanations and Joe Neisendorfer arranged for a semester leave of absence during which I wrote the first draft. Because it will please them, let me acknowledge two grandlings, Alana (1995) and Erika (1998). Their antics provided welcome diversion from the regimen imposed on my writing.

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Index

The book develops the theory of secondary cohomology operations for singular cohomology theory. The author develops the subject in terms of elementary constructions from general homotopy theory. Among many applications considered are the Hopf invariant one theorem (for all primes p, including $p = 2$), Browder's theorem on higher Bockstein operations, and cohomology theory of Massey-Peterson fibrations.

Numerous examples and exercises help readers to gain a working knowledge of the theory. A summary of more advanced parts of the core material is included in the first chapter. Prerequisite is basic algebraic topology, including the Steenrod operations.

The book is written for graduate students and research mathematicians interested in algebraic topology and can be used for self-study or as a textbook for an advanced course on the topic.

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