# Problems in Operator Theory 

## Y. A. Abramovich <br> C. D. Aliprantis

Graduate Studies in Mathematics<br>Volume 5 I

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To the Memory of our Parents

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## Foreword

This book contains complete solutions to the more than six hundred exercises in the authors' book: An Invitation to Operator Theory, American Mathematical Society, 2002. The problems have been spread over eleven chapters following the format of that book. Each problem is identified by a triplet of numbers $x . y . z ; x$ designates the chapter, $y$ the section, and $z$ the exercise. For instance, Problem 3.4.7 indicates Exercise 7 in Section 4 of Chapter 3.

All solutions are based on the material covered in the text with frequent references to the results in the text. For example, a reference to Theorem 5.9 refers to Theorem 5.9 and a reference to Example 6.21 refers to Example 6.21 in the book An Invitation to Operator Theory. We have added an extra amount of material to many solutions in order to make this book as selfcontained as possible.

This problem book will be beneficial to students only if they use it "properly," that is to say, if students look at a solution of a problem only after trying very hard to solve the problem. Students will do themselves great injustice by reading a solution without any prior attempt on the problem. It should be a real challenge to students to produce solutions which are different from the ones presented here.

Due to the extra material incorporated into the problems, the book can be used as a companion supplement to any text used for the standard functional analysis graduate courses. In addition, this solution book can be used as a reference not only for mathematical subjects but also for other disciplines that rely on functional analytic or measure theoretic techniques.

We would like to express our most sincere thanks to all people who made constructive comments and corrections regarding the text and the problems. Special thanks are due to Professors Arkady Kitover and Vladimir Troitsky who read the solutions and made numerous suggestions and corrections. A final thank you goes to Arlene O'Sean, the AMS Copy Editor, for her excellent job in editing the manuscript.
Y. A. Abramovich and C. D. Aliprantis

Indianapolis and West Lafayette, May 2002

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