A Scrapbook of Complex Curve Theory

SECOND EDITION

C. Herbert Clemens

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Preface to the Second Edition

The *Scrapbook*, originally published over twenty years ago, eventually was allowed to go out of print by the original publisher. I am very pleased that the American Mathematical Society is now making it available to a new generation of readers. I am indeed grateful.

The book is an impressionistic journey through the classical subject of complex curves, that is, compact Riemann surfaces. Curve theory, as with most other central areas of complex algebraic geometry, has continued to advance, notably in areas such as the theory of vector bundles on curves and in our understanding of the moduli stacks of curves. But the basics of the theory touched on in the *Scrapbook* have weathered a test of time much longer than the twenty-some years since the first edition and are likely to stand the test of many more decades of scrutiny. For this reason, I saw no need to rewrite the book for this new edition. However, the available literature has grown immensely. So I have taken the opportunity to point out a few more sources in an addendum to the original list of references.

Herb Clemens
Columbus, Ohio
September 2002

Preface

This is a book of "impressions" of a journey through the theory of complex algebraic curves. It is neither self-contained, balanced, nor particularly tightly organized. As with any notebook made on a journey, what appears is that which strikes the writer's fancy. Some topics appear because of their compelling intrinsic beauty. Others are left out because, for all their importance, the traveler found them boring or was too dull or lazy to give them their due.

Looking back at the end of the journey, one can see that a common theme in fact does emerge, as is so often the case; that theme is the theory of theta functions. In fact very much of the material in the book is preparation for our study of the final topic, the so-called Schottky problem. More than once, in fact, we tear ourselves away from interesting topics leading elsewhere and return to our main route.

Some of the subjects are extremely elementary. In fact, we begin with some musings in the vicinity of secondary-school algebra. Later, on occasion and without much warning, we jump into some fairly deep water. Our intent is to struggle with some deep topics in much the same way that a beginning researcher might, using whatever tools we have at hand or can grab somehow or other. Sometimes we use no background material and do everything in detail; sometimes we use some of the heaviest of modern machinery. We hope to motivate further study or, preferably, further discussion with an expert in the field. In short, our aim is to motivate and stimulate mathematical activity rather than to present a finished product, and our point of view is romantic rather than rigorous.

The material treated here was originally brought together for a Summer Course of the Italian National Research Council held in Cortona, Italy, in 1976. It comes from so many sources that adequate acknowledgment would be difficult. The treatment of real two-dimensional geometries of constant curvature comes from Cartan's classic text on Riemannian geometry; several items concerning the arithmetic of curves are borrowed from Serre's lovely book, *A Course in Arithmetic*; Manin's beautiful theorem on rational points of elliptic curves given in Chapter Two was explained to the author by A. Beauville; some of the theta identities in Chapter Three are lifted from the famous analysis text of Whittaker and Watson; and the construction of the level-two moduli space for elliptic curves was motivated by David Mumford's way of viewing the moduli space of curves of a fixed genus. The discussion of the Jacobian variety in Chapter Four leans heavily on work of Joseph Lewittes, and the discussion of the Schottky problem comes from work of Accola, Farkas, Igusa, and Rauch. But perhaps the author's greatest debt is to Phillip Griffiths, through whom he came to enjoy this subject.

The author also wishes to thank Sylvia M. Morris, Mathematics Department of the University of Utah, for preparing the manuscript, and Toni W. Bunker, of the same department, for preparing the figures.

Herbert Clemens
Salt Lake City, Utah

Notation

Most of the notation used in this book is quite standard, for example,

$$\mathbb{Z} = \text{ring of integers},$$
$$\mathbb{Q} = \text{field of rational numbers},$$
$$\mathbb{R} = \text{field of real numbers},$$
$$\mathbb{C} = \text{field of complex numbers}.$$

Each of the six chapters is divided into sections, for instance, Chapter Three has Sections 3.1, 3.2, etc. Equations are numbered consecutively within chapters—(3.1), (3.2), etc.—as are the figures.

Square brackets will be used to enclose matrices and are also used later in the book in expressions involving theta functions with characteristic, for example, $\theta[{1 \atop 1}](u; \tau)$

When there are complicated exponents, the exp form of the exponential is used with the convention $\exp\{x\} = e^x$.

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This fine book by Herb Clemens quickly became a favorite of many complex algebraic geometers when it was first published in 1980. It has been popular with novices and experts ever since. It is written as a book of "impressions" of a journey through the theory of complex algebraic curves. Many topics of compelling beauty occur along the way. A cursory glance at the subjects visited reveals an apparently eclectic selection, from conics and cubics to theta functions, Jacobians, and questions of moduli. By the end of the book, the theme of theta functions becomes clear, culminating in the Schottky problem.

The author's intent was to motivate further study and to stimulate mathematical activity. The attentive reader will learn much about complex algebraic curves and the tools used to study them. The book can be especially useful to anyone preparing a course on the topic of complex curves or anyone interested in supplementing his/her reading.

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