

A Scrapbook of Complex Curve Theory

SECOND EDITION

C. Herbert Clemens

**Graduate Studies
in Mathematics**

Volume 55



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Second Edition

C. Herbert Clemens

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American Mathematical Society
Providence, Rhode Island

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Contents

Preface to the Second Edition	vii
Preface	ix
Notation	xi
Chapter One • Conics	
1.1. Hyperbola Shadows	1
1.2. Real Projective Space, The “Unifier”	5
1.3. Complex Projective Space, The Great “Unifier”	7
1.4. Linear Families of Conics	9
1.5. The Mystic Hexagon	11
1.6. The Cross Ratio	13
1.7. Cayley’s Way of Doing Geometries of Constant Curvature	17
1.8. Through the Looking Glass	20
1.9. The Polar Curve	22
1.10. Perpendiculars in Hyperbolic Space	26
1.11. Circles in the K -Geometry	30
1.12. Rational Points on Conics	33
Chapter Two • Cubics	
2.1. Inflection Points	37
2.2. Normal Form for a Cubic	39
2.3. Cubics as Topological Groups	42
2.4. The Group of Rational Points on a Cubic	45
2.5. A Thought about Complex Conjugation	50
2.6. Some Meromorphic Functions on Cubics	51
2.7. Cross Ratio Revisited, A Moduli Space for Cubics	52
2.8. The Abelian Differential on a Cubic	53
2.9. The Elliptic Integral	55
2.10. The Picard–Fuchs Equation	58
2.11. Rational Points on Cubics over \mathbb{F}_p	62
2.12. Manin’s Result: The Unity of Mathematics	65
2.13. Some Remarks on Serre Duality	69

Chapter Three • Theta Functions	
3.1. Back to the Group Law on Cubics	73
3.2. You Can't Parametrize a Smooth Cubic Algebraically	75
3.3. Meromorphic Functions on Elliptic Curves	78
3.4. Meromorphic Functions on Plane Cubics	82
3.5. The Weierstrass p -Function	85
3.6. Theta-Null Values Give Moduli of Elliptic Curves	89
3.7. The Moduli Space of "Level-Two Structures" on Elliptic Curves ...	92
3.8. Automorphisms of Elliptic Curves	95
3.9. The Moduli Space of Elliptic Curves	96
3.10. And So, By the Way, We Get Picard's Theorem	98
3.11. The Complex Structure of \mathcal{M}	100
3.12. The j -Invariant of an Elliptic Curve	102
3.13. Theta-Nulls as Modular Forms	106
3.14. A Fundamental Domain for Γ_2	109
3.15. Jacobi's Identity	111
Chapter Four • The Jacobian Variety	
4.1. Cohomology of a Complex Curve	113
4.2. Duality	116
4.3. The Chern Class of a Holomorphic Line Bundle	118
4.4. Abel's Theorem for Curves	122
4.5. The Classical Version of Abel's Theorem	127
4.6. The Jacobi Inversion Theorem	131
4.7. Back to Theta Functions	132
4.8. The Basic Computation	134
4.9. Riemann's Theorem	136
4.10. Linear Systems of Degree g	138
4.11. Riemann's Constant	139
4.12. Riemann's Singularities Theorem	142
Chapter Five • Quartics and Quintics	
5.1. Topology of Plane Quartics	147
5.2. The Twenty-Eight Bitangents	150
5.3. Where Are the Hyperelliptic Curves of Genus 3?	155
5.4. Quintics	158
Chapter Six • The Schottky Relation	
6.1. Prym Varieties	161
6.2. Riemann's Theta Relation	164
6.3. Products of Pairs of Theta Functions	167
6.4. A Proportionality Theorem Relating Jacobians and Pryms	168
6.5. The Proportionality Theorem of Schottky–Jung	173
6.6. The Schottky Relation	174
References	181
Additional References	183
Index	185

Preface to the Second Edition

The *Scrapbook*, originally published over twenty years ago, eventually was allowed to go out of print by the original publisher. I am very pleased that the American Mathematical Society is now making it available to a new generation of readers. I am indeed grateful.

The book is an impressionistic journey through the classical subject of complex curves, that is, compact Riemann surfaces. Curve theory, as with most other central areas of complex algebraic geometry, has continued to advance, notably in areas such as the theory of vector bundles on curves and in our understanding of the moduli stacks of curves. But the basics of the theory touched on in the *Scrapbook* have weathered a test of time much longer than the twenty-some years since the first edition and are likely to stand the test of many more decades of scrutiny. For this reason, I saw no need to rewrite the book for this new edition. However, the available literature has grown immensely. So I have taken the opportunity to point out a few more sources in an addendum to the original list of references.

Herb Clemens
Columbus, Ohio
September 2002

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Preface

This is a book of “impressions” of a journey through the theory of complex algebraic curves. It is neither self-contained, balanced, nor particularly tightly organized. As with any notebook made on a journey, what appears is that which strikes the writer’s fancy. Some topics appear because of their compelling intrinsic beauty. Others are left out because, for all their importance, the traveler found them boring or was too dull or lazy to give them their due.

Looking back at the end of the journey, one can see that a common theme in fact does emerge, as is so often the case; that theme is the theory of theta functions. In fact very much of the material in the book is preparation for our study of the final topic, the so-called Schottky problem. More than once, in fact, we tear ourselves away from interesting topics leading elsewhere and return to our main route.

Some of the subjects are extremely elementary. In fact, we begin with some musings in the vicinity of secondary-school algebra. Later, on occasion and without much warning, we jump into some fairly deep water. Our intent is to struggle with some deep topics in much the same way that a beginning researcher might, using whatever tools we have at hand or can grab somehow or other. Sometimes we use no background material and do everything in detail; sometimes we use some of the heaviest of modern machinery. We hope to motivate further study or, preferably, further discussion with an expert in the field. In short, our aim is to motivate and stimulate mathematical activity rather than to present a finished product, and our point of view is romantic rather than rigorous.

The material treated here was originally brought together for a Summer Course of the Italian National Research Council held in Cortona, Italy, in 1976. It comes from so many sources that adequate acknowledgment would be difficult. The treatment of real two-dimensional geometries of

constant curvature comes from Cartan's classic text on Riemannian geometry; several items concerning the arithmetic of curves are borrowed from Serre's lovely book, *A Course in Arithmetic*; Manin's beautiful theorem on rational points of elliptic curves given in Chapter Two was explained to the author by A. Beauville; some of the theta identities in Chapter Three are lifted from the famous analysis text of Whittaker and Watson; and the construction of the level-two moduli space for elliptic curves was motivated by David Mumford's way of viewing the moduli space of curves of a fixed genus. The discussion of the Jacobian variety in Chapter Four leans heavily on work of Joseph Lewittes, and the discussion of the Schottky problem comes from work of Accola, Farkas, Igusa, and Rauch. But perhaps the author's greatest debt is to Phillip Griffiths, through whom he came to enjoy this subject.

The author also wishes to thank Sylvia M. Morris, Mathematics Department of the University of Utah, for preparing the manuscript, and Toni W. Bunker, of the same department, for preparing the figures.

Herbert Clemens
Salt Lake City, Utah

Notation

Most of the notation used in this book is quite standard, for example,

\mathbb{Z} = ring of integers,

\mathbb{Q} = field of rational numbers,

\mathbb{R} = field of real numbers,

\mathbb{C} = field of complex numbers.

Each of the six chapters is divided into sections, for instance, Chapter Three has Sections 3.1, 3.2, etc. Equations are numbered consecutively within chapters—(3.1), (3.2), etc.—as are the figures.

Square brackets will be used to enclose matrices and are also used later in the book in expressions involving theta functions with characteristic, for example, $\theta\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right](u; \tau)$

When there are complicated exponents, the \exp form of the exponential is used with the convention $\exp\{x\} = e^x$.

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Index

- Abelian varieties, 134, 175
- Abel's theorem, 82, 122, 127
- Absolute, the, 17, 28
- Accola, x, 175
- Affine-coordinates, 40
- Affine set, 8
- Albanese variety, 122
- Analytic continuation, 53
- Analytic manifold, 53
- Andreotti, 145, 159
- Anharmonic lines, 30
- Associativity of cubic group law, 44
- Automorphic form, 104; *see also* Modular forms
- Automorphism, 100
 - of elliptic curve, 94, 95

- Base locus, 176
- Basepoint, 55
- Beauville, x
- Bezout's theorem, 12
- Bitangents, 150
- Blowing up a point, 47
- Branch point, 52, 145; *see also* Ramification point

- Canonical bundle, 140, 158
- Canonical divisor, 140, 152, 175
- Canonical mapping, 151, 175
- Cap product, 124
- Cardinality, 63, 67

- Cartan, x
- Cauchy integral formula, 55
- Cayley, 17
- Čech cochain, 70
- Čech one-cochain, 119
- Cell complex, 113
- Character formula, 63
- Characters, 167
- Chern class, 118, 139
- Chinese remainder theorem, 35
- Circle, 30
- Cohomology groups, 134
- Complex conjugation, 50
- Complex manifold, 66, 69, 70, 105
- Component, 12
- Cone, 6, 18, 176
- Congruence, 64
- Conic, 8, 9, 17
 - through five given points, 11
- Contact, 39, 77
- Coordinates
 - affine, 8
 - homogeneous, 8
- Cotangent bundle, 55, 140
- Cotangent space, 66
- Covering group, 134
- Covering space, 58, 89
- Cremona transformation, 48
- Cross ratio, 13, 18, 28, 52, 91, 93
- Cubic, 12, 37, 73, 82, 89
- Cup-product pairing, 116
- Curvature, 20
 - Euclidean, 31
 - geodesic, 31

- Curve
 - cubic, 95
 - elliptic, 75, 91, 94, 95, 96
- Cusp, 154
- Cusp form, 112

- Deformation, 172
- deRham cohomology, 59, 66
- deRham complex, 66
- Desingularization, 78
- Differential, 54, 89
 - abelian, 53
 - exact, 60
 - holomorphic, 152
 - meromorphic, 137
 - Prym, 163
- Differential equation, 59
- Differential forms, 115, 125
- Divisor, 84
- Dolbeault complex, 66
- Double complex, 70, 115, 120
- Double points, 154
- Dual curve, 24
- Dual mapping, 152

- Eisenstein series, 87
- Ellipse, 1
- Elliptic integral, 55
- Euclid's fifth postulate, 17
- Euler characteristic, 55, 101, 145, 154
- Euler's formula, 23, 38

- Family of curves, 147
- Farkas, x , 161
- Fibered product, 52
- Finitely generated abelian group, 50
- Fourier coefficients, 136
- Fourier expansion, 78
- Fourier series, 89, 136, 166
- Fourier transform, 107
- Framing, 94, 95
- Frobenius mapping, 67
- Function
 - meromorphic, 51, 78, 80, 82, 137
 - rational, 75
 - theta, 73, 106, 132, 159, 164
 - trigonometric, 85

- Fundamental domain, 80, 96, 109, 110, 134
- Fundamental group, 161

- Gauss map, 151, 155
- Genus, 114, 149
- Geodesic, 20, 27
- Geometry
 - of constant curvature, 17
 - differential, 54
 - plant, 15
 - Riemannian, 17, 33
- Grassmann variety, 153
- Green's theorem, 56
- Group of rational points, 45
- Gunning, 115, 118, 132

- Hessian curve, 39, 152
- Hürzbruch, 125
- Hodge theory, 115
- Homogeneous equation, 5
- Hyperbola, 1
- Hyperbolic geometry, 21, 26, 32
- Hyperelliptic curve, 151, 155

- Igusa, x
- Implicit function theorem, 54
- Infinity, 40, 50
- Inflection points, 37
- Intersection pairing, 102, 162
- Invariant, 14
 - global, 66
 - local, 66
- Involution, 162, 175
- Isometries, 167
 - group of, 21
- Isomorphism class, 93, 95, 97

- Jacobi inversion theorem, 131, 136
- Jacobian matrix, 65, 153
- Jacobian variety, 113, 123, 127
- Jacobi's identity, 111
- j -Invariant, 102, 105

- Kähler manifold, 66, 115, 125
- K -Geometry, 30

- Kodaira, 145, 158

 Lattices, 165, 170
 Laurent series, 86, 88
 Law of the sine, 13
 Lefschetz duality, 122
 Lefschetz fixed-point theorem, 65
 Lefschetz number, 65
 Lewittes, 144
 Lie group, 131
 Linear fractional transformation, 95
 Linear systems, 138, 176
 Line bundles, 113, 142
 Line integral, 56
 Lorentz transformations, 22

 Manin, x , 65
 Mayer, 145, 159
 Metric, 120
 Euclidean, 32
 induced, 19
 Modular forms, 73, 87, 105, 106, 109, 112
 Moduli space, 53, 91, 96, 101, 145
 Monodromy group, 92
 Mordell's theorem, 50
 Multiplicity of a point, 142
 Mumford, x , 134
 Mystic hexagon, 11, 26

 Nondegenerate plane curve, 24
 Nonsingularity, 147
 Normalization, 156

 Orthonormal basis, 165

 Parabola, 1
 Parametrization, 75
 rational, 76
 Pencil, 176
 Period, 59
 Period matrix, 164, 168, 174
 Perpendicularity, 6, 27
 Picard–Fuchs equation, 58, 64, 68
 Picard's theorem, 98
 Picard variety, 115, 122
 Poincaré dual, 116
 Poincaré mapping, 123, 129, 142
 Point
 infinitely near, 78
 singular, 76, 77
 Poisson summation formula, 89, 108
 Polar coordinates, 56
 Polar curves, 22
 Polar mapping, 24, 29, 37
 Polar of a point, 25
 Pontryagin product, 124, 131
 Power-series expansion, 62
 Projection, 176
 Projective line, 18
 Projective plane,
 complex, 8
 real, 148
 Projective set, 8
 Projective space, 103, 132, 151
 complex, 7
 real, 5
 Projectivization, 8
 Prym varieties, 161, 171, 178

 Quadratic form, 133
 Quadrics, 176
 Quartics, 147
 Quintics, 158

 Ramification, 89
 Ramification point, 43
 Rational points, 33, 62
 Rauch, x
 Regular singular points, 61
 Residue, 59
 Resultant, 9
 Riemann, 133
 Riemann relation, 58, 78, 117, 163
 Riemann–Roch theorem, 68, 85, 87, 95,
 128, 136, 151
 Riemann's constant, 137, 139
 Riemann sphere, 80
 Riemann's singularities theorem, 142, 159,
 178
 Riemann's theorem, 136
 Riemann's theta relation, 111, 165, 167
 Riemann surface, 147, 175

 Scalar product, 165

- Schottky, 161
- Schottky–Jung, 173, 178
- Schottky relation, 164, 174, 178
- Serre, vi, 87, 105, 132, 145
- Serre duality, 68, 69
- Sheaf, 67
 - of holomorphic functions, 114
- Sheaf cohomology, 66, 113
- Skew-Hermitian matrix, 126
- Spectral sequence, 115
- Spencer, 145
- Spherical distance, 19
- Spherical geometry, 26
- Stereographic projection, 26, 33, 42, 51, 76
- Surface integral, 56
- Symmetric product, 25, 52, 131
- Symplectic basis, 102, 114, 133

- Tangent cone, 77, 160
- Tangent line, 154

- Tangent spaces, 150
- Tate, 44, 46, 73
- Theta characteristics, 140, 143, 155, 157, 158, 171, 177
- Theta functions, 132
- Theta-null, 89, 106, 109
- Tjurin, 158
- Topological group, 43
- Torus, 43
- Transversality, 65

- Unbranched double covering, 160, 161
- Universal covering spaces, 133
- Upper half-plane, 97, 103

- van der Monde determinant, 10

- Wedge algebra, 124
- Weierstrass p -function, 58, 85, 87

This fine book by Herb Clemens quickly became a favorite of many complex algebraic geometers when it was first published in 1980. It has been popular with novices and experts ever since. It is written as a book of "impressions" of a journey through the theory of complex algebraic curves. Many topics of compelling beauty occur along the way. A cursory glance at the subjects visited reveals an apparently eclectic selection, from conics and cubics to theta functions, Jacobians, and questions of moduli. By the end of the book, the theme of theta functions becomes clear, culminating in the Schottky problem.

The author's intent was to motivate further study and to stimulate mathematical activity. The attentive reader will learn much about complex algebraic curves and the tools used to study them. The book can be especially useful to anyone preparing a course on the topic of complex curves or anyone interested in supplementing his/her reading.

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