

A Course in Algebra

E. B. Vinberg

**Graduate Studies
in Mathematics**

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in Mathematics

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Preface

My motivation for writing this book came from teaching a two year course in algebra at the Mathematical College of the Independent University of Moscow in 1992–1994. The students' enthusiasm and a relatively small class allowed me to keep the level of presentation higher than it is usually done at the Mechanics and Mathematics Department of Moscow State University, and to touch on a number of subjects beyond a regular university course. However, in writing this book I used my experience in teaching at Moscow State University, and so the final version of the book is only partially related to the course given at the Independent University.

Chapters 1–7 and part of Chapter 8 more or less correspond to the first year algebra course at the Mechanics and Mathematics Department of Moscow State University. The remaining chapters cover, and, in fact, substantially exceed the second year algebra course. These chapters are intended mainly for students specializing in algebra.

Note that Chapter 7 is devoted to geometry of Euclidean, affine, and projective spaces. However, this chapter should not be viewed as an exposition of geometry; rather, it describes the algebraic approach to geometry.

In the first four chapters I tried to make the presentation sufficiently detailed to be suitable for a reader such as a mathematics freshman at Moscow State University. (However, the language of sets and maps is used from the very beginning without any explanations.) In later chapters I allowed myself to skip details that can be easily reconstructed, since I believe that a reader should gradually acquire mathematical culture.

There are almost no technically difficult proofs in this book. Following my point view on mathematics, I tried to replace calculations and difficult

deductions with conceptual proofs. Some readers may find this style hard, but the efforts spent in absorbing new ideas will pay off when the students start solving problems not discussed in this book.

For the English edition, the bibliography at the end of the book was revised. It is certainly not complete and, to some extent, arbitrary, but I believe the reader might find it helpful.

I am grateful to all current and former members of the Chair of Higher Algebra at the Mechanics and Mathematics Department of Moscow State University who helped me to shape my approach to teaching algebra.

In the English translation a number of misprints and inaccuracies were corrected and some explanations added.

Moscow, November 2002

E. B. Vinberg

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