

# Concise Numerical Mathematics

Robert Plato

Graduate Studies  
in Mathematics

Volume 57



American Mathematical Society

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Robert Plato

Translated by  
Richard Le Borne  
Sabine Le Borne

Graduate Studies  
in Mathematics

Volume 57



American Mathematical Society  
Providence, Rhode Island

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Originally published in the German language by Friedr. Vieweg & Sohn Verlagsgesellschaft mbH, D-65189 Wiesbaden, Germany, under the title “Robert Plato: Numerische Mathematik kompakt. 1. Auflage (1st edition)”. © Friedr. Vieweg & Sohn Verlagsgesellschaft mbH, Braunschweig/Wiesbaden, 2000.

Translated from the original German edition by Richard Le Borne and Sabine Le Borne.

2000 *Mathematics Subject Classification*. Primary 65-00, 65-01, 65B05, 65B15, 65Dxx, 65Fxx, 65G50, 65H05, 65H10, 65H17, 65K10, 65Lxx, 65Q05, 65T40, 65T50, 65Y20, 49M15, 49M20.

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### Library of Congress Cataloging-in-Publication Data

Plato, Robert, 1962–

[Numerische Mathematik kompakte. English]

Concise numerical mathematics / Robert Plato ; translated by Richard Le Borne, Sabine Le Borne.

p. cm. – (Graduate studies in mathematics, ISSN 1065-7339 ; v. 57)

Includes bibliographical references and index.

ISBN 0-8218-2953-X (acid-free paper) ISBN 0-8218-3414-2 (softcover)

1. Numerical analysis. I. Title. II. Series.

QA297.P56713 2003  
519.4-dc21

2002033010

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10 9 8 7 6 5 4 3 2 1 08 07 06 05 04 03

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# Preface to the English Edition

This textbook is a translation of *Numerische Mathematik kompakt* published by Vieweg Verlag, with only a few changes: most of the references to German monographs appearing in the German edition are replaced by references to English written textbooks, some new publications are added, and a few errors have been corrected. A webpage for this book is maintained on the AMS website at:

<http://www.ams.org/bookpages/gsm-57>.

I would like to thank Richard C. Le Borne and Sabine Le Borne (they are both at Tennessee Technological University, Cookeville) for the careful translation and for a very efficient and pleasant cooperation. I am grateful to several colleagues and students for suggestions on the German edition which are considered to some extent in this edition. Moreover, I express thanks to the Christian-Albrechts University for travel support in connection with this translation and for giving me the opportunity during the last two years to hold introductory courses on numerical mathematics for students of mathematics, engineering, and computer sciences. These courses were based on the Vieweg edition of this textbook. Additionally, I would like to thank Edward Dunne from the American Mathematical Society for several helpful suggestions and his patience.

Finally two comments on frequently used notations. The symbol  $\Delta$  marks the end of an example, a remark, or an algorithm, and the symbol ‘— « —’ is used as a ditto mark.

Berlin, Germany

October 2002

Robert Plato

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# Preface to the German Edition

This textbook emerged from my two-semester courses in numerical mathematics which I have repeatedly taught since 1997 at the Technical University of Berlin. These lectures were attended primarily by students of mathematics and to a smaller extent by students of physics and computer sciences.

In its current form this textbook addresses students and graduates of mathematics, computer sciences, and natural and engineering sciences. In a concise form several basic topics of numerical mathematics with important applications are treated:

- interpolation, fast Fourier transform and integration,
- direct and iterative solution of linear systems of equations,
- iterative methods for nonlinear systems of equations,
- numerical solution of initial and boundary value problems for ordinary differential equations,
- eigenvalue problems for matrices,
- approximation theory and computer arithmetic.

For the purpose of conciseness, the numerical solution of partial differential equations as well as nonlinear optimization is not treated here.

The intention of this textbook is to consider topics in an elementary and clear form, thereby only requiring a basic knowledge of calculus and linear

algebra. Moreover, many of the presented methods are illustrated by figures. For numerous algorithms, practical and relevant work estimates are given and pseudo codes are provided that can be used for implementations. The approximately 120 exercises presented with different levels of difficulty proved to be good in the courses.

I used preliminary versions of this textbook as lecture notes for my two-semester courses in numerical mathematics, each course meeting four hours per week. The first six chapters were always considered in the first part, and Chapters 7–13 were treated in the second part of these lectures. In a one-semester course in numerical mathematics, an alternative is to consider numerical integration (Chapter 6) only briefly, and then to present the foundations of one-step methods for the numerical solution of initial value problems for ordinary differential equations (Chapter 7) and relaxation methods for the iterative solution of linear systems of equations (Chapter 10).

For this textbook there exists an online service which is available under <http://www.math.tu-berlin.de/numerik/plato/viewegbuch>. It contains hints for the presented exercises and MATLAB routines for some of the pseudo codes presented in the textbook. Moreover, this online service will contain a list of possibly necessary corrections. Comments on this textbook can be sent to my email address, [plato@math.tu-berlin.de](mailto:plato@math.tu-berlin.de).

I would like to thank my colleagues Prof. Dr. R. D. Grigorieff and Dipl. Math. Etienne Emmrich for several suggestions which are considered in this textbook to a large extent. I am indebted to Till Tantau and Olivier Pfeiffer and some other participants of the lectures for several minor but important improvements. Finally I would like to thank Prof. Dr. Chuck Groetsch, Prof. Dr. Martin Hanke-Bourgeois and Prof. Dr. Hans-Jürgen Reinhardt for their support of this book project and Ulrike Schmickler-Hirzebruch from the publisher Vieweg for her pleasant cooperation.

Berlin, May 2000

Robert Plato

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