Cartan for Beginners: Differential Geometry via Moving Frames and Exterior Differential Systems

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Contents

Preface

In this book, we use moving frames and exterior differential systems to study geometry and partial differential equations. These ideas originated about a century ago in the works of several mathematicians, including Gaston Darboux, Edouard Goursat and, most importantly, Elie Cartan. Over the years these techniques have been refined and extended; major contributors to the subject are mentioned below, under "Further Reading".

The book has the following features: It concisely covers the classical geometry of surfaces and basic Riemannian geometry in the language of moving frames. It includes results from projective differential geometry that update and expand the classic paper [**69**] of Griffiths and Harris. It provides an elementary introduction to the machinery of exterior differential systems (EDS), and an introduction to the basics of G-structures and the general theory of connections. Classical and recent geometric applications of these techniques are discussed throughout the text.

This book is intended to be used as a textbook for a graduate-level course; there are numerous exercises throughout. It is suitable for a one-year course, although it has more material than can be covered in a year, and parts of it are suitable for one-semester course (see the end of this preface for some suggestions). The intended audience is both graduate students who have some familiarity with classical differential geometry and differentiable manifolds, and experts in areas such as PDE and algebraic geometry who want to learn how moving frame and EDS techniques apply to their fields.

In addition to the geometric applications presented here, EDS techniques are also applied in CR geometry (see, e.g., [**98**]), robotics, and control theory (see [**55, 56, 129**]). This book prepares the reader for such areas, as well as for more advanced texts on exterior differential systems, such as [**20**], and papers on recent advances in the theory, such as [**58, 117**].

Overview. Each section begins with geometric examples and problems. Techniques and definitions are introduced when they become useful to help solve the geometric questions under discussion. We generally keep the presentation elementary, although advanced topics are interspersed throughout the text.

In Chapter 1, we introduce moving frames via the geometry of curves in the Euclidean plane $\mathbb{E}^2$. We define the Maurer-Cartan form of a Lie group G and explain its use in the study of submanifolds of G-homogeneous spaces. We give additional examples, including the equivalence of holomorphic mappings up to fractional linear transformation, where the machinery leads one naturally to the Schwarzian derivative.

We define exterior differential systems and jet spaces, and explain how to rephrase any system of partial differential equations as an EDS using jets. We state and prove the Frobenius system, leading up to it via an elementary example of an overdetermined system of PDE.

In Chapter 2, we cover traditional material—the geometry of surfaces in three-dimensional Euclidean space, submanifolds of higher-dimensional Euclidean space, and the rudiments of Riemannian geometry—all using moving frames. Our emphasis is on local geometry, although we include standard global theorems such as the rigidity of the sphere and the Gauss-Bonnet Theorem. Our presentation emphasizes finding and interpreting differential invariants to enable the reader to use the same techniques in other settings.

We begin Chapter 3 with a discussion of Grassmannians and the Plücker embedding. We present some well-known material (e.g., Fubini's theorem on the rigidity of the quadric) which is not readily available in other textbooks. We present several recent results, including the Zak and Landman theorems on the dual defect, and results of the second author on complete intersections, osculating hypersurfaces, uniruled varieties and varieties covered by lines. We keep the use of terminology and results from algebraic geometry to a minimum, but we believe we have included enough so that algebraic geometers will find this chapter useful.

Chapter 4 begins our multi-chapter discussion of the Cartan algorithm and Cartan-Kähler Theorem. In this chapter we study constant coefficient homogeneous systems of PDE and the linear algebra associated to the corresponding exterior differential systems. We define tableaux and involutivity of tableaux. One way to understand the Cartan-Kähler Theorem is as follows: given a system of PDE, if the linear algebra at the infinitesimal level "works out right" (in a way explained precisely in the chapter), then existence of solutions follows.

In Chapter 5 we present the Cartan algorithm for linear Pfaffian systems, a very large class of exterior differential systems that includes systems of PDE rephrased as exterior differential systems. We give numerous examples, including many from Cartan's classic treatise [**31**], as well as the isometric immersion problem, problems related to calibrated submanifolds, and an example motivated by variation of Hodge structure.

In Chapter 6 we take a detour to discuss the classical theory of characteristics, Darboux's method for solving PDE, and Monge-Ampère equations in modern language. By studying the exterior differential systems associated to such equations, we recover the sine-Gordon representation of pseudospherical surfaces, the Weierstrass representation of minimal surfaces, and the one-parameter family of non-congruent isometric deformations of a surface of constant mean curvature. We also discuss integrable extensions and Bäcklund transformations of exterior differential systems, and the relationship between such transformations and Darboux integrability.

In Chapter 7, we present the general version of the Cartan-Kähler Theorem. Doing so involves a detailed study of the integral elements of an EDS. In particular, we arrive at the notion of a Kähler-regular flag of integral elements, which may be understood as the analogue of a sequence of well-posed Cauchy problems. After proving both the Cartan-Kähler Theorem and Cartan's test for regularity, we apply them to several examples of non-Pfaffian systems arising in submanifold geometry.

Finally, in Chapter 8 we give an introduction to geometric structures (G-structures) and connections. We arrive at these notions at a leisurely pace, in order to develop the intuition as to why one needs them. Rather than attempt to describe the theory in complete generality, we present one extended example, path geometry in the plane, to give the reader an idea of the general theory. We conclude with a discussion of some recent generalizations of G-structures and their applications.

There are four appendices, covering background material for the main part of the book: linear algebra and rudiments of representation theory, differential forms and vector fields, complex and almost complex manifolds, and a brief discussion of initial value problems and the Cauchy-Kowalevski Theorem, of which the Cartan-Kähler Theorem is a generalization.

Layout. All theorems, propositions, remarks, examples, etc., are numbered together within each section; for example, Theorem 1.3.2 is the second numbered item in section 1.3. Equations are numbered sequentially within each chapter. We have included hints for selected exercises, those marked with the symbol ⊚ at the end, which is meant to be suggestive of a life preserver.

Further Reading on EDS. To our knowledge, there are only a small number of textbooks on exterior differential systems. The first is Cartan's classic text [**31**], which has an extraordinarily beautiful collection of examples, some of which are reproduced here. We learned the subject from our teacher Bryant and the book by Bryant, Chern, Griffiths, Gardner and Goldschmidt [**20**], which is an elaboration of an earlier monograph [**19**], and is at a more advanced level than this book. One text at a comparable level to this book, but more formal in approach, is [**156**]. The monograph [**70**], which is centered around the isometric embedding problem, is similar in spirit but covers less material. The memoir [**155**] is dedicated to extending the Cartan-Kähler Theorem to the C^∞ setting for hyperbolic systems, but contains an exposition of the general theory. There is also a monograph by Kähler [**89**] and lectures by Kuranishi [**97**], as well the survey articles [**66, 90**]. Some discussion of the theory may be found in the differential geometry texts [**142**] and [**145**].

We give references for other topics discussed in the book in the text.

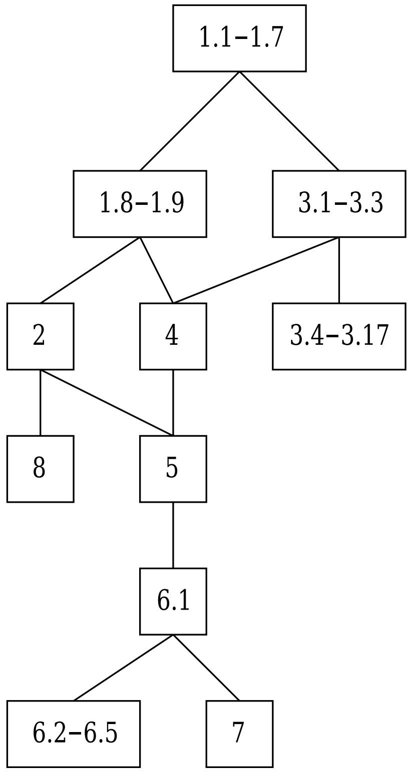
History and Acknowledgements. This book started out about a decade ago. We thought we would write up notes from Robert Bryant's Tuesday night seminar, held in 1988–89 while we were graduate students, as well as some notes on exterior differential systems which would be more introductory than [**20**]. The seminar material is contained in §8.6 and parts of Chapter 6. Chapter 2 is influenced by the many standard texts on the subject, especially [**43**] and [**142**], while Chapter 3 is influenced by the paper [**69**]. Several examples in Chapter 5 and Chapter 7 are from [**31**], and the examples of Darboux's method in Chapter 6 are from [**63**]. In each case, specific attributions are given in the text. Chapter 7 follows Chapter III of [**20**] with some variations. In particular, to our knowledge, Lemmas 7.1.10 and 7.1.13 are original. The presentation in §8.5 is influenced by [**11**], [**94**] and unpublished lectures of Bryant.

The first author has given graduate courses based on the material in Chapters 6 and 7 at the University of California, San Diego and at Case Western Reserve University. The second author has given year-long graduate courses using Chapters 1, 2, 4, 5, and 8 at the University of Pennsylvania and Université de Toulouse III, and a one-semester course based on Chapters 1, 2, 4 and 5 at Columbia University. He has also taught one-semester

undergraduate courses using Chapters 1 and 2 and the discussion of connections in Chapter 8 (supplemented by [**141**] and [**142**] for background material) at Toulouse and at Georgia Institute of Technology, as well as one-semester graduate courses on projective geometry from Chapters 1 and 3 (supplemented by some material from algebraic geometry), at Toulouse, Georgia Tech. and the University of Trieste. He also gave more advanced lectures based on Chapter 3 at Seoul National University, which were published as [**107**] and became a precursor to Chapter 3. Preliminary versions of Chapters 5 and 8 respectively appeared in [**104, 103**].

We would like to thank the students in the above classes for their feedback. We also thank Megan Dillon, Phillipe Eyssidieux, Daniel Fox, Sung-Eun Koh, Emilia Mezzetti, Joseph Montgomery, Giorgio Ottaviani, Jens Piontkowski, Margaret Symington, Magdalena Toda, Sung-Ho Wang and Peter Vassiliou for comments on the earlier drafts of this book, and Annette Rohrs for help with the figures. The staff of the publications division of the AMS—in particular, Ralph Sizer, Tom Kacvinsky, and our editor, Ed Dunne—were of tremendous help in pulling the book together. We are grateful to our teacher Robert Bryant for introducing us to the subject. Lastly, this project would not have been possible without the support and patience of our families.

Dependence of Chapters



Suggested uses of this book:

- a year-long graduate course covering moving frames and exterior differential systems (chapters 1–8);
- a one-semester course on exterior differential systems and applications to partial differential equations (chapters 1 and 4–7);
- a one-semester course on the use of moving frames in algebraic geometry (chapter 3, preceded by part of chapter 1);
- a one-semester beginning graduate course on differential geometry (chapters 1, 2 and 8).

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This book is an introduction to Cartan's approach to differential geometry. Two central methods in Cartan's geometry are the theory of exterior differential systems and the method of moving frames. The book presents thorough and modern treatments of both subjects, including their applications to classic and contemporary problems.

The book begins with the classical geometry of surfaces and basic Riemannian geometry in the language of moving frames, along with an elementary introduction to exterior differential systems. Key concepts are developed incrementally, with motivating examples leading to definitions, theorems, and proofs.

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