

Functional Analysis

An Introduction

Yuli Eidelman
Vitali Milman
Antonis Tsoolomitis

**Graduate Studies
in Mathematics**

Volume 66



American Mathematical Society

Functional Analysis

An Introduction

This page intentionally left blank

Functional Analysis

An Introduction

Yuli Eidelman

Vitali Milman

Antonis Tsoolomitis

Graduate Studies
in Mathematics

Volume 66



American Mathematical Society
Providence, Rhode Island

Editorial Board

Walter Craig
Nikolai Ivanov
Steven G. Krantz
David Saltman (Chair)

2000 *Mathematics Subject Classification*. Primary 46-01, 47-01; Secondary 46Axx, 46Bxx, 46Cxx, 46Hxx, 47Axx, 47Bxx.

For additional information and updates on this book, visit
www.ams.org/bookpages/gsm-66

Library of Congress Cataloging-in-Publication Data

Eidelman, Yuli, 1955–

Functional analysis : an introduction / Yuli Eidelman, Vitali Milman, Antonis Tsolomitis.

p. cm. — (Graduate studies in mathematics, ISSN 1065-7339 ; v. 66)

Includes bibliographical references and indexes.

ISBN 0-8218-3646-3 (alk. paper)

1. Functional analysis. 2. Hilbert algebras. 3. Operator theory. I. Milman, Vitali D., 1939–
II. Tsolomitis, Antonis. III. Title. IV. Series.

QA320.E38 2004
515'.7—dc22

2004057393

Copying and reprinting. Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy a chapter for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication is permitted only under license from the American Mathematical Society. Requests for such permission should be addressed to the Acquisitions Department, American Mathematical Society, 201 Charles Street, Providence, Rhode Island 02904-2294, USA. Requests can also be made by e-mail to reprint-permission@ams.org.

© 2004 by the American Mathematical Society. All rights reserved.

The American Mathematical Society retains all rights
except those granted to the United States Government.

Printed in the United States of America.

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.

Visit the AMS home page at <http://www.ams.org/>

10 9 8 7 6 5 4 3 2 1 09 08 07 06 05 04

Contents

Preface *xi*

Introduction *xiii*

Part I. Hilbert Spaces and Basic Operator Theory 1

1 Linear spaces; normed spaces; first examples 3

- 1.1 Linear spaces 3
- 1.2 Normed spaces; first examples 7
 - 1.2a Hölder's inequality 9
 - 1.2b Minkowski's inequality 11
- 1.3 Topological and geometric notions 12
- 1.4 Quotient normed space 14
- 1.5 Completeness; completion 16
- 1.6 Exercises 21

2 Hilbert spaces 25

- 2.1 Basic notions; first examples 25
 - 2.1a Cauchy-Schwartz inequality and the Hilbertian norm 26
 - 2.1b Bessel's inequality 29
 - 2.1c Complete systems 29

2.1d	Gram-Schmidt orthogonalization procedure; orthogonal bases	30
2.1e	Parseval's identity	32
2.2	Projection; orthogonal decompositions	34
2.2a	Separable case	34
2.2b	The distance from a point to a convex set	35
2.2c	Orthogonal decomposition	35
2.3	Linear functionals	37
2.3a	Linear functionals in a general linear space	37
2.3b	Bounded linear functionals	38
2.3c	Bounded linear functionals in a Hilbert space	40
2.3d	An example of a non-separable Hilbert space	41
2.4	Exercises	41
3	The dual space	47
3.1	The Hahn-Banach theorem and its first consequences	47
3.1a	Corollaries of the Hahn-Banach theorem	48
3.2	Examples of dual spaces	51
3.3	Exercises	52
4	Bounded linear operators	55
4.1	Completeness of the space of bounded linear operators	55
4.2	Examples of linear operators	57
4.3	Compact operators	59
4.3a	Compact sets	59
4.3b	The space of compact operators	62
4.4	Dual operators	63
4.5	Operators of finite rank	65
4.5a	Compactness of the integral operator in L_2	66
4.6	Convergence in the space of bounded operators	67
4.7	Invertible operators	68
4.8	Exercises	69

5	Spectrum. Fredholm theory of compact operators	75
5.1	Classification of spectrum	75
5.2	Fredholm theory of compact operators	77
5.3	Exercises	83
6	Self-adjoint operators	87
6.1	General properties	87
6.2	Self-adjoint compact operators	89
6.2a	Spectral theory	89
6.2b	Minimax principle	92
6.2c	Applications to integral operators	94
6.3	Order in the space of self-adjoint operators	96
6.3a	Properties of the ordering	96
6.4	Projection operators	100
6.4a	Properties of projections in linear spaces	100
6.4b	Orthoprojections	101
6.5	Exercises	103
7	Functions of operators; spectral decomposition	105
7.1	Spectral decomposition	108
7.1a	The main inequality	109
7.1b	Construction of the spectral integral	110
7.2	Hilbert theorem	111
7.3	Spectral family and spectrum of self-adjoint operators	114
7.4	Simple spectrum	116
7.5	Exercises	117
Part II. Basics of Functional Analysis		119
8	Spectral theory of unitary operators	121
8.1	Spectral properties of unitary operators	121
8.2	Exercises	126

- 9 The fundamental theorems and the basic methods 127**
 - 9.1 Auxiliary results 128
 - 9.2 The Banach open mapping theorem 131
 - 9.3 The closed graph theorem 132
 - 9.4 The Banach-Steinhaus theorem 135
 - 9.5 Bases in Banach spaces 139
 - 9.6 Linear functionals; the Hahn-Banach theorem 142
 - 9.7 Separation of convex sets 146
 - 9.8 The Eberlain-Schmulian theorem 152
 - 9.9 Extremal points; the Krein-Milman theorem 153
 - 9.10 Exercises 159

- 10 Banach algebras 167**
 - 10.1 Preliminaries 167
 - 10.2 Gelfand's theorem on maximal ideals 171
 - 10.3 Analytic functions 172
 - 10.4 Gelfand map; the space of maximal ideals 176
 - 10.4a The space of maximal ideals 179
 - 10.5 Radicals 181
 - 10.6 Involutions; the Gelfand-Naimark theorem 184
 - 10.7 Application to spectral theory 189
 - 10.8 Application to a generalized limit and combinatorics 193
 - 10.9 Exercises 196

- 11 Unbounded self-adjoint and symmetric operators in H 203**
 - 11.1 Basic notions and examples 203
 - 11.2 More properties of symmetric operators 210
 - 11.3 The spectrum $\sigma(A)$ 211
 - 11.4 Elements of the "graph method" 215
 - 11.5 Cayley transform; spectral decomposition 216
 - 11.6 Symmetric and self-adjoint extensions of a symmetric operator 221
 - 11.7 Exercises 225

A Solutions to exercises	227
A.1 Solutions to the exercises of Chapter 1	227
A.2 Solutions to the exercises of Chapter 2	235
A.3 Solutions to the exercises of Chapter 3	250
A.4 Solutions to the exercises of Chapter 4	254
A.5 Solutions to the exercises of Chapter 5	263
A.6 Solutions to the exercises of Chapter 6	270
A.7 Solutions to the exercises of Chapter 7	277
A.8 Solutions to the exercises of Chapter 8	279
A.9 Solutions to the exercises of Chapter 9	282
A.10 Solutions to the exercises of Chapter 10	296
A.11 Solutions to the exercises of Chapter 11	309
Bibliography	311
Symbols index	313
Subject index	317

This page intentionally left blank

Preface

The goal of this textbook is to provide an introduction to the language and methods of functional analysis, including chapters on Hilbert spaces, operator theory, basic theorems and methods of abstract functional analysis and a few applications of these methods to Banach algebras and the theory of unbounded self-adjoint operators.

The text represents notes for a series of two courses (12 to 14 weeks each; 3 hours weekly and 1 hour of exercises and discussions for both courses):

- (i) Introduction to Hilbert spaces and operator theory (Chapters 1–7),
- (ii) Introduction to functional analysis (Chapters 8–11).

I gave these courses for many years at Tel Aviv University and also, once in 1995, at the Ohio State University (OSU), Columbus. That one time at OSU was a very lucky time for me, because my then Ph.D. student Antonis Tsolomitis worked on my very rough notes of the lectures and suggested creating this book. He was indispensable in his rôle in our joint effort, and the book would not have come to publication without his agreement to join me in writing it.

Another stroke of luck came my way with the huge wave of emigration of Russian mathematicians in the 1990's. Among them was Dr. Yuli Eidelman, who at the start of his career here had time to assist me in the courses. He prepared exercises and material suitable for discussions. So, the final textbook is the result of the efforts of all three of us, Tony, Yuli and myself.

A few words about functional analysis and some necessary background: one very important goal of mathematics is to develop a language, a so-called “mathematical language”. Firstly, it is needed for the very precise exchange of thoughts, and secondly—and not less importantly—we develop the terminology which fixes our understanding and catches new mathematical observations and laws. This continuously developing terminology “compresses” achievements of the previous stage of development of mathematics into “spoken language”. Once deep theorems become a language and we no longer (need to) think of them as theorems, it helps us to “free” our minds in preparation for a new portion of mathematics.

(For example: we say “a linear space of dimension n ”, but a deep theorem stands behind this sentence. The notion of “dimension” is a theorem which students study in a linear algebra course.)

The most important rôle of functional analysis was to develop a mathematical language. Functional analysis became the language of twentieth century mathematics (more precisely its part called analysis) and theoretical physics. Even articles on popular science and science fiction books use this language and talk about “operators” and their “spectrum”.

To teach students to speak in this language is the main goal of this textbook. Numerous theorems (sometimes very short in their proofs) should help us in the end to feel comfortable with new notions, to get used to them, to speak new words without painful efforts to recall what they mean. They should become a part of the reader’s mathematical culture.

We would like to emphasize that we took special care to be brief and not to overload the students (and other readers) with the enormous amount of information available on the subject. Over the years we have checked that the amount of mathematics presented in this course is absorbable in a year’s study and provides the basis for future reading.

Vitali Milman
Tel Aviv, Israel

Introduction

As mentioned in the Preface, the text corresponds to two courses (12 to 14 weeks each; 3 hours weekly and 1 hour of exercises and discussions for both courses):

- (i) Introduction to Hilbert spaces and operator theory (Chapters 1–7),
- (ii) Introduction to functional analysis (Chapters 8–11).

These courses require only the knowledge from any first course in Linear Algebra. However, for the second course it would be useful if the reader had some knowledge of measure theory.

The book does not contain any “additional” chapters. Material, although important but which cannot be condensed into these two courses in the time available, without overloading the reader’s ability to digest new notions and facts, is not included.

The reader may look at the bibliography for books that complement the material of this text. Moreover, one can see there other possibilities for presenting the same results.

In Chapter 1 we introduce linear spaces and normed spaces and we give some first, but important, examples. The spaces $L_p[a, b]$ are introduced through the completion of the continuous functions with the L_p -norm in order to avoid requiring the knowledge of measure theory from the reader.

In the second chapter, Hilbert spaces are introduced and we prove basic facts about them. Linear functionals are also introduced in this

chapter, which closes with a natural example of a non-separable Hilbert space.

Chapter 3 discusses the notion of the dual Banach spaces. The Hahn-Banach theorem is stated here without proof and with the standard corollaries, needed for the rest of the course. The Hahn-Banach theorem is proved in the second course, in Chapter 9.

Chapter 4 introduces the bounded linear operators, the compact operators, the dual operators and the invertible operators. We also discuss a different kind of convergence in the space of bounded operators. Here we state the open mapping theorem and we postpone its proof until Chapter 9 (the Banach-Steinhaus theorem is not stated before Chapter 9 since it can certainly be avoided until then).

Chapter 5 is on spectral theory for the general operator. The classification of spectrum is discussed here as well as the development of Fredholm theory.

In Chapters 6 and 7 we focus our attention on the spectral theory of self-adjoint operators. The spectral integral is also given here.

Although Chapter 8, on the spectral theory of unitary operators, would fit more naturally into the first part of this course, we advise postponing dealing with it until Chapter 11, on unbounded symmetric operators. One reason is that the concepts of spectral theory and the spectral integral are not easy to absorb at first, and it is worthwhile returning later to basics to tackle them. Another reason is that the spectral theory of unbounded self-adjoint operators and the Cayley transform naturally begin with the understanding of unitary operators.

Chapter 9 contains the general, more classical results which form the base and the methods of functional analysis. Besides the main theorems of the theory and the central notions of weak and weak* topology, we selected a number of “branch” results, with a twofold goal in mind. We want to demonstrate how the method works and we want to get used to the notion and language of functional analysis; and we also want to use this opportunity to introduce additional important notions and enlarge the picture.

We selected the two remaining chapters of the course for the following reasons. First, we want to show that, by adding natural structure

to the basic notion of Banach space, we quickly derive deep, rich and very concrete analytic consequences. So, Chapter 10 provides an introduction to Gelfand's beautiful theory of Banach algebras.

At this stage, the reader may get the feeling that the whole of functional analysis is about "good", "well-organized" objects. To remove this misapprehension and to show that the methods of functional analysis can deal with less "good" objects, e.g., unbounded operators, we consider unbounded symmetric operators and present the spectral theory of self-adjoint (unbounded) operators in Chapter 11.

There are many books written on this subject. Some of them are strictly textbooks, and some are more of a monograph type. We mention some of them in the bibliography at the end of the book. We would like to mention especially the books [GGK03] by I. Gohberg, S. Goldberg and M.A. Kaashoek and [AKR78] by A.B. Antonevich, P.N. Knyazev and Ya.V. Radyno, which were helpful for us when preparing some of the exercises. We recommend these books for additional reading.

Y. Eidelman, Tel Aviv, Israel

V. Milman, Tel Aviv, Israel

A. Tsolomitis, Samos, Greece

This page intentionally left blank

Bibliography

- [AkhGL] N.I. Akhiezer, I.M. Glazman, *Theory of linear operators in Hilbert space*, Translation: Merlynd Nestell, Dover, 1993.
- [AKR78] A. B. Antonevich, P. N. Knyazev and Ya. V. Radyno, *Problems and exercises on functional analysis*, Vysheisha Shkola, Minsk, 1978 (Russian).
- [Bol90] Béla Bollobás, *Linear analysis, an introductory course*, Cambridge Mathematical Textbooks, 1990.
- [BSU96] Y. M. Berezansky, Z. G. Sheftel and G. F. Us, *Functional analysis*, Birkhauser Verlag, Basel, 1996.
- [Con90] John B. Conway, *A course in functional analysis*, second edition, Springer 1990, Graduate Texts in Mathematics, number 96.
- [DS58-63] N. Danford and J. T. Schwartz, *Linear operators*, Interscience Publishers, New York, Part 1: 1958, Part 2: 1963.
- [Edw94] R.E. Edwards, *Functional analysis, theory and applications*, Dover, 1994.
- [GGK03] I. Gohberg, S. Goldberg and M. A. Kaashoek, *Basic classes of linear operators*, Birkhauser Verlag, Basel, 2003.
- [KF70] A.N. Kolmogorov and S.V. Fomin, *Introductory real analysis*, Dover, 1970.
- [Kre78] E. Kreyszig, *Introductory functional analysis with applications*, John Wiley & Sons, 1978.
- [LS85] L.A. Lusternik and V.J. Sobolev, *Elements of functional analysis*, Gordon and Breach Science Publishers, 1985.

- [Nik92] N.K. Nikol'skij (Ed.), *Functional analysis I*, Encyclopedia of Mathematical Sciences, vol. 19, Springer-Verlag, 1992.
- [Rud91] Walter Rudin, *Functional analysis*, second edition, McGraw-Hill, 1991.
- [SteSh] E.M. Stein, R. Shakarchi, *Fourier analysis, an introduction*, Princeton University Press, 2003.
- [Yos80] Kôsaku Yoshida, *Functional analysis*, sixth edition, Springer-Verlag, 1980, Grundlehren der Mathematischen Wissenschaften, number 123.
- [Zim90] Robert J. Zimmer, *Essential results of functional analysis*, Chicago Lecture Notes in Mathematics, 1990.

Symbols index

- Miscellaneous symbols
- $(a_n) * (b_n)$, convolution of sequences, 169
- $*$, involution, 185
- $L \hookrightarrow H$, L is a subspace of H , 34
- $L \oplus M$, orthogonal sum of the subspaces L and M , 36
- X^* , the dual space of X , 47
- X^{**} , the second dual space of X , 50
- $\Delta_A(\lambda)$, 212
- \xrightarrow{w} , convergence in the weak topology, 68
- $f \otimes y$, tensor product, 65
- $x * y$, convolution of x with y , 169
- $x \perp y$, x is orthogonal to y , 27
- $\langle x, y \rangle$, the inner product of x and y , 25
- $[x]$, coset with representative x , 5
- $\|\cdot\|^*$, the dual norm, 38
- $\|K\|_{\text{op}}$, norm of the integral operator with kernel K , 57
- $\|f\|_p$, the L_p -norm of f , i.e., the quantity $\left(\int_a^b |f|^p\right)^{1/p}$, 16
- $\|x\|$, norm of the vector x , 7
- $\|x\|_1$, norm of the element x of ℓ_1 , i.e., the number $\sum |x_n|$, 8
- $\|x\|_p$, norm of the element x of ℓ_p , i.e., the number $(\sum |x_n|^p)^{1/p}$, 8
- A
- \mathcal{A} , a Banach algebra, 167
- \mathcal{A} , net, 61
- \overline{A} , closure of the set A , 13

\sqrt{A} , the non-negative root of the positive operator A , 98

A^* , the dual operator of A , 63

$\overset{\circ}{A}$, the interior of the set A , 128

algspan, the algebraic span, 185

$\arg x$, the argument of x , i.e., the quantity $x/|x|$, 51

C

c , set of convergent sequences, 4

c_0 , set of null sequences, 4

c_0^* , the dual of c_0 , 51

$C_2[a, b]$, continuous functions f on $[a, b]$ with norm $\|f\| = \left(\int_a^b |f|^2\right)^{1/2}$, 27

\mathbb{C}^n , complex n -dimensional space, 3

$C_p[a, b]$, the set of all continuous functions on $[a, b]$ equipped with the norm $\|f\|_p = \left(\int_a^b |f|^p\right)^{1/p}$, 16, 21

$C_{\mathbb{R}}$, the space of all continuous and bounded functions on \mathbb{R} , 83

D

δ_a , the Dirac functional, 37

$\mathcal{D}(E)$, unit ball of the space E , 14

$\dim E$, dimension of the linear space E , 5

Δ_λ , the range of the operator $T - \lambda I$, 77

$D_r(x_0)$, open ball with radius r and center x_0 , 12

$d(x, y)$, distance between the points x and y , 7

E

\hat{E} , the completion of E , 19

E^\sharp , the space of linear functionals on E , 37

ε -net, 61

E/E_1 , quotient space, 6

E_λ , the spectral orthoprojection $e_\lambda(A)$ (of the operator A), 109

$e_\lambda(t)$, increasing family of step functions with the discontinuity at λ , 108

Extr K , the set of the extremal points of the set K , 154

F

F_\perp , (dual) annihilator, 49

F^c , complement of the set F , 13

$f_n \searrow f$, the sequence f_n decreases and converges to f , 105

G

$\Gamma(A)$, the graph of the operator A , 132

I

$\text{Im}A$, image of the operator A , 4

$I[x, y]$, interval with endpoints x and y , 13

K

$\ker A$, kernel of the operator A ,
4
 $\overset{c}{K}$, the kernel of the set K , 129

L

L^\perp , the orthogonal complement
of the subspace L , 36
 ℓ_1 , space of sequences (x_n) with
 $\sum |x_n|$ finite, 8
 $L_2[a, b]$, the completion of
 $C_2[a, b]$, 21, 27
 $\ell_2(w)$, the ℓ_2 space with
weighted inner prod-
uct by w , 42
 $\ell_2(\mathbb{Z})$, the space of
bi-directional se-
quences $(x_n)_{n \in \mathbb{Z}}$ with
 $\sum_{n \in \mathbb{Z}} |x_n|^2 < \infty$, 69
 L^\perp , the annihilator of L , 49
 ℓ_∞ , set of bounded sequences,
4
 ℓ_p , space of all sequences (x_n)
with $\sum |x_n|^p$ finite, 8
 L_p^* , the dual of the space L_p , 52
 ℓ_p^* , the dual space of ℓ_p , 51
 $L_p[a, b]$, the completion of
 $C_p[a, b]$, 21
 ℓ_p^n , the n -dimensional space
with the $\|x\|_p$ -norm,
11
 $L(X, Y)$, the linear space of
bounded operators
from X to Y , 55

$L(X \mapsto Y)$, the linear space
of bounded operators
from X to Y , 55

M

\mathcal{M} , the space of all maximal ide-
als of a Banach algebra,
179
 $\mathcal{M}(\mathcal{A})$, the space of all maximal
ideals of the Banach al-
gebra \mathcal{A} , 179

P

$P_L x$, the projection of x on the
subspace L , 34
 $p_M(x)$, the value at the vector x
of the Minkowski func-
tional of the set M , 146

Q

\mathbb{Q} , rational numbers, 17

R

\mathbb{R} , real numbers, 3
 $\text{rad}(\mathcal{A})$, the radical of the alge-
bra \mathcal{A} , 181
 \mathbb{R}^n , real n -dimensional space,
3
 $R_p[a, b]$, the functions on $[a, b]$
whose p -th power is
Riemann integrable,
16
 $\rho(x, L)$, the distance from x to
the subspace L , 34

S

s , set of all sequences, 4

s^* , set of sequences of finite support, 4

$\sigma(A)$, the spectrum of the operator A , 75

$\sigma_c(A)$, the continuous spectrum of the operator A , 76

$\sigma_p(A)$, the point spectrum of the operator A , 75

$\sigma_r(A)$, the residual spectrum of the operator A , 76

$\mathcal{S}(E)$, unit sphere of the space E , 14

$\text{span } M$, linear span of the set M , 5

T

T_λ , abbreviation for the operator $T - \lambda I$, 77

W

$w(F)$, the weak topology generated by functionals in F , 153

X

$x + E$, coset with representative x , 5

Subject index

A

Alaoglu, 149
algebra
 semisimple, 181
algebraic span, 185
almost periodic function, 188
analytic functions, 172
Arzelá, 60

B

Baire category, 128
Baire-Hausdorff, 128
Banach
 algebra, 167
 limit, 163
 space, 17, 127
Banach-Steinhaus, 67, 135, 136
band matrix, 70
basis, 140
 of a normed space, 31
 of space, 5
Bessel inequality, 29, 31

biorthogonal

 functionals, 139
 system, 50

Birkhoff's theorem, 155

bounded

 functional, 38
 set, 14
 variation, 161

C

category, 128

Cauchy

 integral, 173
 sequence, 16

Cauchy-Schwartz inequality,
 10, 26

Cayley transform, 216

center of a set, 129

centrally symmetric, 14

closed

 graph operator, 132, 133
 set, 12

closure, 13

closure of operator, 133
codimension, 6
compact operators, 59
compact set, 59
complete, 18
 space, 17
 system, 29, 30
completion, 19, 21, 117
conjugate, 184
continuous
 functional, 38
 spectrum, 76, 212
convex, 14, 35, 144
 set, 13
convolution of sequences, 169
cosets, 5

D

decomposition of the identity,
 109
dense set, 13
dimension of space, 5
Dirac measure, 157
direct decomposition, 101
Dirichlet kernel, 139
distance to a set, 35
domain of operator, 203
dual
 norm, 47
 operator, 63
 space, 47, 51

E

Eberlain-Schmulian, 152
eigenvalue, 75, 211
 multiplicity, 76

eigenvector, 212
embedding operator, 60
equicontinuous, 60
equivalent norms, 8, 52
essential involution, 188
extremal
 point, 153, 154
 set, 153

F

field, 171
finite
 rank, 65
 rank operators, 65
 support sequence, 4
first category set, 128
Fisher, 93
Fourier transform, 121
Fredholm
 first theorem, 81
 second theorem, 81
 theory, 77
 third theorem, 82

G

Gelfand, 167, 171
 map, 176
 theorem, 171
Gelfand-Mazur, 176
Gelfand-Naimark theorem,
 186
generalized
 Cauchy-Schwartz inequality,
 96
 nilpotent, 182

- generator of simple spectrum,
116
- Goldstein, 151
- Gram-Schmidt, 30
- graph
method, 215
of operator, 132
- H
- Haar system, 162
- Hahn-Banach, 47, 142
- Heine-Borel, 107
- Hilbert, 3, 25
space, 27, 28
- Hilbert space, 33
- Hilbert-Courant, 93
- Hilbert-Schmidt
first theorem, 90
second theorem, 94
- hilbertian norm, 27
- Hölder's inequality, 9
- Hörmander, 134
- I
- ideal, 170
- image, 4, 131
- index of defect, 206
- inequality
Cauchy-Schwartz, 10, 26
generalized, 96
Hölder, 9
integral form, 10
Minkowski, 11
integral form, 12
- inner product, 25
- integral representation, 173
- interval, 13
- invariant subspace, 87, 90
- inverse operator, 68
- invertible
element of a Banach algebra, 170
operator, 68
- involution, 184
essential, 188
- isometry, 19, 116
into, 19
- isomorphism, 4
- J
- James, 153
- K
- kernel, 4, 15, 130
function, 57
of a set, 129
of an operator, 57
- Krein-Milman, 153, 154
- L
- limit along an ultrafilter, 193
- linear
map, 3, 4
space, 3
span, 5
- linear functional, 37
positive, 188
- linearly
dependent
set, 5
vectors, 5
- independent, 30
relative to subspace, 6

linearly independent vectors,
5

Livshič, 129

Lorantian polynomial, 122

M

majorizing element, 172

maximal ideal, 171

measure

Dirac, 157

probability, 156

strongly regular, 156

Mercer's theorem, 95, 96

minimal system, 139

minimax principle, 93

Minkowski's

functional, 146, 147

inequality, 11

multiplicity of the eigenvalue,
76

N

non-separable, 41

norm, 7

convergence, 67

equivalent, 8

stronger, 8

normed space, 7, 8, 14

norming set, 164

nowhere dense, 128

O

open

ball, 12

radius, 12

map, 69

set, 12

operator, 55

compact, 59

domain, 203

graph of, 132

index of defect, 206

isometry, 116

of finite rank, 65

self-adjoint, 87, 204

symmetric, 87, 204

unitary, 116

unitary equivalent, 117,
224

order of self-adjoint operators,
96

orthogonal

complement, 36

decomposition, 34, 35

orthonormal system, 28

orthoprojection, 101

P

pairing, 64

Parallelogram Law, 27, 88

Parseval's identity, 32

perfectly convex set, 129

point spectrum, 75

Polya, 138

polydisc, 199

positive linear functional, 188

precompact

set, 59

space, 60

probability measure, 156

projection, 34

operators, 100

spectral family, 124

proper ideal, 170, 171
 Pythagorean theorem, 27

Q

quotient
 normed space, 14
 space, 5, 16, 19

R

radical, 181
 Ramsey theorem, 194
 rank of an operator, 65
 reducing subspace, 122
 reflexive space, 50
 regular point, 75, 173
 relatively compact set, 59
 residual spectrum, 76, 212
 Riesz representation, 40

S

scalar product, 25
 Schauder
 basis, 140
 system, 162
 Schauder basis, 140
 second category set, 128
 self-adjoint
 element, 185
 operator, 87, 204
 semilinearity, 25
 seminorm, 15
 semisimple algebra, 181
 separable space, 31
 separation of convex sets, 147
 set
 center of, 129
 compact, 59

 first category, 128
 kernel of, 129
 perfectly convex, 129
 precompact, 59
 relatively compact, 59
 second category, 128
 shift operator, 58
 simple spectrum, 116
 generator, 116
 singular point, 174
 space
 Banach, 17, 127
 of bounded operators, 55
 of maximal ideals, 176
 reflexive, 50
 spectral
 decomposition, 108, 217
 family, 109
 family of projections, 124
 integral, 110
 properties, 121
 theory, 75
 spectral decomposition, 108
 spectrum, 75, 211
 spectrum point, 75
 strong convergence, 67
 stronger norm, 8
 strongly
 continuous function, 161
 holomorphic function,
 161
 regular measure, 156
 sublinear function, 142
 subspace, 4
 invariant, 87

symmetric

- algebra, 185
- extension, 204
- kernels, 94
- operator, 87, 204

T

tensor product, 66

theorem

- Alaoglu, 149
- Banach-Steinhaus, 136
- Cauchy, 173
- Eberlain-Schmulian, 152
- Fredholm
 - first, 81
 - second, 81
 - third, 82
- Gelfand, 171
- Gelfand-Naimark, 186
- Goldstein, 151
- Hörmander, 134
- Hahn-Banach, 47, 142
- James, 153
- Krein-Milman, 153, 154
- Liouville, 175
- Mercer, 95
- Polya, 138
- Ramsey, 194
- Wiener, 178

total set, 48

triangle inequality, 7

U

uniform convergence, 67

uniformly bounded, 60

unit

ball, 14

sphere, 14, 48

unitary operator, 121

unitary operators, 116
equivalent, 116, 224

V

Volterra operator, 82

W

weak

convergence, 68

topology, 148

weak* topology, 148

weakly

continuous function, 161

holomorphic function,
161

Weierstraß, 30

Wiener

algebra, 169, 178

theorem, 178

Z

Zabreïko, 135

Zorn's lemma, 154

The book "*Functional analysis, An introduction*" was typeset using L^AT_EX₂ ϵ . The indexes were generated with MAKEINDEX and the bibliography with BIBT_EX₈. The PostScript file was generated with DVIPS and for PDF conversions we used GHOSTSCRIPT 8.00. The final DVI file was 1.4 Mbytes. Editing was entirely done with GNU-EMACS on Linux 2.4 on x86. The book was published by the $\mathcal{A}\mathcal{M}\mathcal{S}$ in Autumn 2004.

This page intentionally left blank

Titles in This Series

- 66 **Yuli Eidelman, Vitali Milman, and Antonis Tsolomitis**, Functional analysis, An introduction, 2004
- 65 **S. Ramanan**, Global calculus, 2004
- 64 **A. A. Kirillov**, Lectures on the orbit method, 2004
- 63 **Steven Dale Cutkosky**, Resolution of singularities, 2004
- 62 **T. W. Körner**, A companion to analysis: A second first and first second course in analysis, 2004
- 61 **Thomas A. Ivey and J. M. Landsberg**, Cartan for beginners: Differential geometry via moving frames and exterior differential systems, 2003
- 60 **Alberto Candel and Lawrence Conlon**, Foliations II, 2003
- 59 **Steven H. Weintraub**, Representation theory of finite groups: algebra and arithmetic, 2003
- 58 **Cédric Villani**, Topics in optimal transportation, 2003
- 57 **Robert Plato**, Concise numerical mathematics, 2003
- 56 **E. B. Vinberg**, A course in algebra, 2003
- 55 **C. Herbert Clemens**, A scrapbook of complex curve theory, second edition, 2003
- 54 **Alexander Barvinok**, A course in convexity, 2002
- 53 **Henryk Iwaniec**, Spectral methods of automorphic forms, 2002
- 52 **Ilka Agricola and Thomas Friedrich**, Global analysis: Differential forms in analysis, geometry and physics, 2002
- 51 **Y. A. Abramovich and C. D. Aliprantis**, Problems in operator theory, 2002
- 50 **Y. A. Abramovich and C. D. Aliprantis**, An invitation to operator theory, 2002
- 49 **John R. Harper**, Secondary cohomology operations, 2002
- 48 **Y. Eliashberg and N. Mishachev**, Introduction to the h -principle, 2002
- 47 **A. Yu. Kitaev, A. H. Shen, and M. N. Vyalyi**, Classical and quantum computation, 2002
- 46 **Joseph L. Taylor**, Several complex variables with connections to algebraic geometry and Lie groups, 2002
- 45 **Inder K. Rana**, An introduction to measure and integration, second edition, 2002
- 44 **Jim Agler and John E. M^cCarthy**, Pick interpolation and Hilbert function spaces, 2002
- 43 **N. V. Krylov**, Introduction to the theory of random processes, 2002
- 42 **Jin Hong and Seok-Jin Kang**, Introduction to quantum groups and crystal bases, 2002
- 41 **Georgi V. Smirnov**, Introduction to the theory of differential inclusions, 2002
- 40 **Robert E. Greene and Steven G. Krantz**, Function theory of one complex variable, 2002
- 39 **Larry C. Grove**, Classical groups and geometric algebra, 2002
- 38 **Elton P. Hsu**, Stochastic analysis on manifolds, 2002
- 37 **Hershel M. Farkas and Irwin Kra**, Theta constants, Riemann surfaces and the modular group, 2001
- 36 **Martin Schechter**, Principles of functional analysis, second edition, 2002
- 35 **James F. Davis and Paul Kirk**, Lecture notes in algebraic topology, 2001
- 34 **Sigurdur Helgason**, Differential geometry, Lie groups, and symmetric spaces, 2001
- 33 **Dmitri Burago, Yuri Burago, and Sergei Ivanov**, A course in metric geometry, 2001
- 32 **Robert G. Bartle**, A modern theory of integration, 2001
- 31 **Ralf Korn and Elke Korn**, Option pricing and portfolio optimization: Modern methods of financial mathematics, 2001
- 30 **J. C. McConnell and J. C. Robson**, Noncommutative Noetherian rings, 2001
- 29 **Javier Duoandikoetxea**, Fourier analysis, 2001
- 28 **Liviu I. Nicolaescu**, Notes on Seiberg-Witten theory, 2000
- 27 **Thierry Aubin**, A course in differential geometry, 2001
- 26 **Rolf Berndt**, An introduction to symplectic geometry, 2001

TITLES IN THIS SERIES

- 25 **Thomas Friedrich**, Dirac operators in Riemannian geometry, 2000
- 24 **Helmut Koch**, Number theory: Algebraic numbers and functions, 2000
- 23 **Alberto Candel and Lawrence Conlon**, Foliations I, 2000
- 22 **Günter R. Krause and Thomas H. Lenagan**, Growth of algebras and Gelfand-Kirillov dimension, 2000
- 21 **John B. Conway**, A course in operator theory, 2000
- 20 **Robert E. Gompf and András I. Stipsicz**, 4-manifolds and Kirby calculus, 1999
- 19 **Lawrence C. Evans**, Partial differential equations, 1998
- 18 **Winfried Just and Martin Weese**, Discovering modern set theory. II: Set-theoretic tools for every mathematician, 1997
- 17 **Henryk Iwaniec**, Topics in classical automorphic forms, 1997
- 16 **Richard V. Kadison and John R. Ringrose**, Fundamentals of the theory of operator algebras. Volume II: Advanced theory, 1997
- 15 **Richard V. Kadison and John R. Ringrose**, Fundamentals of the theory of operator algebras. Volume I: Elementary theory, 1997
- 14 **Elliott H. Lieb and Michael Loss**, Analysis, 1997
- 13 **Paul C. Shields**, The ergodic theory of discrete sample paths, 1996
- 12 **N. V. Krylov**, Lectures on elliptic and parabolic equations in Hölder spaces, 1996
- 11 **Jacques Dixmier**, Enveloping algebras, 1996 Printing
- 10 **Barry Simon**, Representations of finite and compact groups, 1996
- 9 **Dino Lorenzini**, An invitation to arithmetic geometry, 1996
- 8 **Winfried Just and Martin Weese**, Discovering modern set theory. I: The basics, 1996
- 7 **Gerald J. Janusz**, Algebraic number fields, second edition, 1996
- 6 **Jens Carsten Jantzen**, Lectures on quantum groups, 1996
- 5 **Rick Miranda**, Algebraic curves and Riemann surfaces, 1995
- 4 **Russell A. Gordon**, The integrals of Lebesgue, Denjoy, Perron, and Henstock, 1994
- 3 **William W. Adams and Philippe Loustaunau**, An introduction to Gröbner bases, 1994
- 2 **Jack Graver, Brigitte Servatius, and Herman Servatius**, Combinatorial rigidity, 1993
- 1 **Ethan Akin**, The general topology of dynamical systems, 1993

The goal of this textbook is to provide an introduction to the methods and language of functional analysis, including Hilbert spaces, Fredholm theory for compact operators, and spectral theory of self-adjoint operators. It also presents the basic theorems and methods of abstract functional analysis and a few applications of these methods to Banach algebras and the theory of unbounded self-adjoint operators.


The text corresponds to material for two semester courses (Part I and Part II, respectively), and it is as self-contained as possible. The only prerequisites for the first part are minimal amounts of linear algebra and calculus. However, for the second course (Part II), it is useful to have some knowledge of topology and measure theory. Each chapter is followed by numerous exercises, whose solutions are given at the end of the book.

ISBN 978-0-8218-3646-0



9 780821 836460

GSM/66

For additional information
and updates on this book, visit 
www.ams.org/bookpages/gsm-66

AMS on the Web
www.ams.org