Recurrence and Topology

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Gail S. Nelson

Graduate Studies in Mathematics
Volume 85

American Mathematical Society

http://dx.doi.org/10.1090/gsm/085

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2000 *Mathematics Subject Classification.* Primary 37-01, 37B20, 37B25, 37B35, 54H20;
Secondary 37C10, 37C15, 37C25, 37C27, 37C50, 37C70, 34D45.

For additional information and updates on this book, visit
www.ams.org/bookpages/gsm-85

Library of Congress Cataloging-in-Publication Data

Alongi, John M., 1971–
Recurrence and topology / John M. Alongi, Gail S. Nelson.
p. cm. — (Graduate studies in mathematics, ISSN 1065-7339 ; v. 85)
Includes bibliographical references and index.
ISBN-13: 978-0-8218-4234-8 (alk. paper)
ISBN-10: 0-8218-4234-X (alk. paper)
1. Recurrent sequences (Mathematics) 2. Point mappings (Mathematics) 3. Topology.
I. Nelson, Gail Susan. II. Title.

QA9.6.A46 2007
515′.24–dc22 2007060754

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10 9 8 7 6 5 4 3 2 1 12 11 10 09 08 07

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Foreword

During the early nineteenth century the French mathematician Simeon Denis Poisson (1781–1840) observed through his study of celestial mechanics that the solutions of a differential equation which exhibit recurrent behavior more general than periodicity play a central role in determining the disposition of all solutions of the system. Since the time of Poisson, mathematicians have sought precisely what it means for a solution of a differential equation to be recurrent.

Recurrence and Topology develops increasingly more general topological modes of recurrence for dynamical systems beginning with fixed points and concluding with chain recurrent points. For each type of recurrence we provide detailed examples arising from explicit systems of differential equations; we establish the general topological properties of the set of recurrent points; and we investigate the possibility of partitioning the set of recurrent points into subsets which are dynamically irreducible. Furthermore, we consider how test functions such as invariant functions, potential functions and Lyapunov functions describe the structures of sets of recurrent points. The text concludes with a statement, proof, and interpretation of the Fundamental Theorem of Dynamical Systems due to Charles Conley (1933–1984).

Recurrence and Topology has a deliberately narrow focus. We treat flows (continuous dynamical systems) rather than maps (discrete dynamical systems) for three reasons. First, flows arise directly and naturally from the differential equations that motivate the subject. Second, the connectedness of orbits for flows sets the stage for a rich interplay between recurrence and topology. Third, chains for flows rely on both distance and time, while chains for maps rely only on distance.

Recurrence and Topology does not treat results, such as the Poincaré-Bendixson Theorem, which are particular to low dimension, nor do we discuss recurrence in the context of measure theory nor recurrence arising from hyperbolicity. However, we hope there is a niche for this text in seminar courses, independent studies, and as a supplement to comprehensive works such as Katok and Hasselblatt [**26**] and Robinson [**37**] which develop many threads within the subject simultaneously. *Recurrence and Topology* examines a single thread and aims to present it in a clear, complete and coherent manner.

While the objective of *Recurrence and Topology* is to integrate existing knowledge, we offer detailed original proofs of a few folklore theorems (such as the fact that a flow's chain components are exactly the connected components of the flow's chain recurrent set). We present competing definitions of certain terms (such as what it means for a set to be topologically transitive with respect to a flow) and argue for the definitions we adopt.

With the work of Stephen Smale (1930–) during the 1960s and the genesis of chaos as a scientific paradigm during the mid-1970s, the field of dynamical systems enjoyed a renaissance. Today it seems appropriate to reflect on advances in the field, to place them in context, and to make them accessible to a broader mathematical audience. We hope that *Recurrence and Topology* is a step in that direction.

Recurrence and Topology is appropriate for mathematics graduate students, though a well-prepared undergraduate might read most of the text with great benefit. We presume differential equations, undergraduate analysis and general topology as background. The texts by Boyce and DiPrima [**10**] Rudin [**38**] and Munkres [**32**] provide sufficient preparation. Certain sections also require advanced linear algebra and complex variables. After referring to manifolds in Section 1.1 for motivation, we do not require the geometry of manifolds until Sections 4.5 and 4.6. Milnor [**31**], Guillemin and Pollack [**17**], and Hicks [**20**] are good references for the relevant material. Regarding our exposition, we emphasize clarity over brevity and apologize to readers who find our presentation excessively detailed.

John Alongi thanks John Franks, Clark Robinson, Bob Welland, Don Saari, Jeff Xia and Keith Burns for teaching him about dynamical systems, topology and differential equations. Thanks to the Department of Mathematics and Computer Science at Pomona College for providing financial support from its Spears account during the 2003–2004 and 2004–2005 academic years. Most of all, thanks to Michael and Joan Alongi, David, Tina and Kristine Alongi, Ryan Beveridge and Tricia Hansen, Alissa Crans, Allison Engel, and Connie and Jim Pemble.

Gail Nelson thanks Steve Kennedy, Sam Patterson, and John Alongi for inviting her to join their research seminar, giving her the opportunity to learn more about dynamical systems. Thanks also to the rest of the members of the Department of Mathematics at Carleton College for their support of this project. Many thanks to her family, Rik, Julie, and Lauren Getty, Doug, Teresa, Thomas, and Anna Bearrood, and especially Edward Nelson, who has always been her role model.

Both authors thank Joseph Borzellino, Karen Brucks, Ezra Buchla, Richard Elderkin, Francesca Fairbrother, John Franks, Barbara Jenkins, Stephen Kennedy, Laurene Lee, Sam Patterson, Ami Radunskaya, Clark Robinson, Anthony W. Sevold and Kathy Sheldon for their assistance in this endeavor. In particular, Ezra Buchla and Anthony W. Sevold rendered the figures in the text. Laurene Lee offered many important suggestions while reading a draft of the text. Both authors also thank the anonymous reviewers of the text as well as the editors and staff of the American Mathematical Society for their thoughtful suggestions. Ed Dunne, Cristin Zannella and Luann Cole of the AMS deserve special recognition in this regard.

Please direct comments and corrections to `jalongi@northwestern.edu` or `gnelson@carleton.edu`.

John M. Alongi
Chicago, IL

Gail S. Nelson
Northfield, MN

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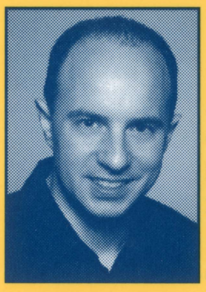

Photograph by Mike Canale

Since at least the time of Poisson, mathematicians have pondered the notion of recurrence for differential equations. Solutions that exhibit recurrent behavior provide insight into the behavior of general solutions. In *Recurrence and Topology*, Alongi and Nelson provide a modern understanding of the subject, using the language and tools of dynamical systems and topology.

Recurrence and Topology develops increasingly more general topological modes of recurrence for dynamical systems beginning with fixed points and concluding with chain recurrent points. For each type of recurrence the text provides detailed examples arising from explicit systems of differential equations; it establishes the general topological properties of the set of recurrent points; and it investigates the possibility of partitioning the set of recurrent points into subsets which are dynamically irreducible. The text includes a discussion of real-valued functions that reflect the structure of the sets of recurrent points and concludes with a thorough treatment of the Fundamental Theorem of Dynamical Systems.

Recurrence and Topology is appropriate for mathematics graduate students, though a well-prepared undergraduate might read most of the text with great benefit.

ISBN 978-0-8218-4234-8

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