Graduate Algebra: Noncommutative View

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Graduate Algebra: Noncommutative View
To the memory of my beloved mother
Ruth Halle Rowen, April 5, 1918 – January 5, 2007
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Introduction

As indicated in the title, this volume is concerned primarily with noncommutative algebraic structures, having grown from a course introducing complex representations of finite groups via the structure of group algebras and their modules. Our emphasis is on algebras, although we also treat some major classes of finite and infinite groups. Since this volume was conceived as a continuation of Volume 1 (Graduate Algebra: Commutative View, Graduate Studies in Mathematics, volume 73), the numeration of chapters starts with Chapter 13, Part IV, and we use the basics of rings and modules developed in Part I of Volume 1 (Chapters 1–3). Nevertheless, Chapters 13–15 and 18 can largely be read independently of Volume 1.

In the last one hundred years there has been a vast literature in noncommutative theory, and our goal here has been to find as much of a common framework as possible. Much of the theory can be cast in terms of representations into matrix algebras, which is our major theme, dominating our treatment of algebras, groups, Lie algebras, and Hopf algebras. A secondary theme is the description of algebraic structures in terms of generators and relations, pursued in the appendices of Chapter 17, and leading to a discussion of free structures, growth, word problems, and Zelmanov's solution of the Restricted Burnside Problem.

One main divergence of noncommutative theory from commutative theory is that left ideals need not be ideals. Thus, the important notion of “principal ideal” from commutative theory becomes cumbersome; whereas the principal left ideal $Ra$ is described concisely, the smallest ideal of a noncommutative ring $QR$ containing an element $a$ includes all elements of the form

$$r_{1,1}ar_{1,2} + \cdots + r_{m,1}ar_{m,2}, \quad \forall r_{i,1}, r_{i,2}, \in R,$$
where $m$ can be arbitrarily large. This forces us to be careful in distinguishing “left” (or “right”) properties from two-sided properties, and leads us to rely heavily on modules.

There are many approaches to structure theory. We have tried to keep our proofs as basic as possible, while at the same time attempting to appeal to a wider audience. Thus, projective modules (Chapter 25) are introduced relatively late in this volume.

The exposition is largely self-contained. Part IV requires basic module theory, especially composition series (Chapter 3 of Volume 1). Chapter 16 draws on material about localization and Noetherian rings from Chapters 8 and 9 of Volume 1. Chapter 17, which goes off in a different direction, requires some material (mostly group theory) given in the prerequisites of this volume. Appendix 17B generalizes the theory of Gröbner bases from Appendix 7B of Volume 1. Chapter 18 has applications to field theory (Chapter 4 of Volume 1).

Parts V and VI occasionally refer to results from Chapters 4, 8, and 10 of Volume 1. At times, we utilize quadratic forms (Appendix 0A) and, occasionally, derivations (Appendix 6B). The end of Chapter 24 draws on material on local fields from Chapter 12. Chapters 25 and 26 require basic concepts from category theory, treated in Appendix 1A.

There is considerable overlap between parts of this volume and my earlier book, Ring Theory (student edition), but the philosophy and organization is usually quite different. In Ring Theory the emphasis is on the general structure theory of rings, via Jacobson’s Density Theorem, in order to lay the foundations for applications to various kinds of rings.

The course on which this book is based was more goal-oriented — to develop enough of the theory of rings for basic representation theory, i.e., to prove and utilize the Wedderburn-Artin Theorem and Maschke’s Theorem. Accordingly, the emphasis here is on semisimple and Artinian rings, with a short, direct proof. Similarly, the treatment of Noetherian rings here is limited mainly to Goldie’s Theorem, which provides most of the non-technical applications needed later on.

Likewise, whereas in Ring Theory we approached representation theory of groups and Lie algebras via ring-theoretic properties of group algebras and enveloping algebras, we focus in Part V of this volume on the actual groups and Lie algebras.

Thanks to Dror Pak for pointing me to the proofs of the hook categories, to Luda Markus-Epstein for material on Stallings foldings, to Alexei Belov for gluing components in the Wedderburn decomposition, and to Sue Montgomery for a description of the current state of the classification of
finite dimensional Hopf algebras. Steve Shnider, Tal Perri, Shai Sarussi, and Luie Polev provided many helpful comments. Again, as with Volume 1, I would like to express special gratitude to David Saltman, in particular for his valuable suggestions concerning Chapter 24 and Chapter 25, and also to Uzi Vishne. Thanks to Sergei Gelfand for having been patient for another two years. And, of course, many thanks to Miriam Beller for much of the technical preparation of the manuscript.

Needless to say, I am deeply indebted to Rachel Rowen, my helpmate, for her steadfast support all of these years.
List of Symbols

Warning: Some notations have multiple meanings.

\[ [G : H] \quad xxiii \]
\[ A_Q \quad xxiv \]
\[ M_n(R) \quad 5 \]
\[ L \neq R, \ Ann_R a \quad 6 \]
\[ e_{ij}, \delta_{ij} \quad 7 \]
\[ R^{op} \quad 15 \]
\[ \text{Cent}(R), \prod R_i, \pi_i, \nu_i \quad 16 \]
\[ \text{Hom}_R(M, N) \quad 19 \]
\[ \text{End}_R(M), \text{End}_R(M)_W, \text{Ann}_R S \quad 20 \]
\[ W^{(n)} \quad 22 \]
\[ \ell_r \quad 25 \]
\[ R[\lambda], R[[\lambda]] \quad 27 \]
\[ [a, b], \mathcal{A}_1(F) \quad 28 \]
\[ \mathcal{A}_n(F) \quad 30 \]
\[ \text{soc}(M), \text{soc}(R) \quad 33 \]
\[ \mathbb{H} \quad 41 \]
\[ S, K \quad 43 \]
\[ L_1L_2, RaR \quad 45 \]
\[ A^2, A^k \quad 49 \]
\[ \text{Jac}(R) \quad 50 \]
\[ N(R) \quad 65 \]
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<td>$D_n, D_\infty, S_n$</td>
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<td>$\bar{\psi}, f \otimes g$</td>
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<td>$\mu: R \times R \to R$</td>
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<td>$S(M), C(V, Q), E(V) \wedge$</td>
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<td>$C[G]$</td>
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<td>$z_G$</td>
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$\lambda$

$P(T), Q(T)$

$I_{\lambda}$

$H_{i,j}, h_{i,j}$

$\text{GL}(n, F), \text{SL}(n, F), \text{D}(n, F), \text{UT}(n, F), \text{UT}(n, F), \text{O}(n, F), \text{U}(n, \mathbb{C})$

$\text{Sp}(n, F), \text{SO}(n, F), \text{PGL}(n, F)$

$G_{g}, G_{e}$

$\Delta, \epsilon$

$\xi, \rho$

$\mathbb{R}$

$\rho \otimes \tau$

$\rho^{G}$

$[ab], [a, b], R^{-}$

$\text{ad}_a, \text{ad}_L, \text{ad}_L H, A_n, B_n, C_n, D_n, gl(n, F), sl(n + 1), F$

$Z(L)$

$N_L(A),$

$L,$

$s, n$

$a[p], L^k, L^{(k)}, L'$

$\text{rad}(L)$

$I^{\perp}$

$e^*_1, \ldots, e^*_n$

$c_\rho$

$\text{Null}(a), a, L_a$

$\text{Null}(N)$

$r_f, \langle f, g \rangle, \langle a, b \rangle,$

$P, S$

$m_{ij}$

$\mathbf{v} > 0; \mathbf{v} \geq 0$

$Lie(G) T(G)_e$

$[x, y, z]$

$S(R, *)$

$J(V, Q)$

$U(L)$
Symbols

$U_q(sl(2, F))$  
$U(J)$  
$A_n, B_n, C_n, D_n, E_n, F_n, G_n$  
$m_{ij}$  
$R \# G$  
$B_n$  
$x_i \mapsto r_i, f(R), \text{id}(R)$  
$h_{alt}, s_t, c_t$  
$\Delta_i f$  
$g_n, E(V)$  
$h_n$  
$\text{id}(ValV)$  
$C\{Y\}_n$  
$I_n(R), c_n(R)$  
$F\{Y, Z\}, \text{id}_2(R)$  
$e(\pi, I), f_I$  
$G(R)$  
$\mathcal{F} \mathcal{J}$  
$\mathcal{F} \mathcal{S} \mathcal{J}$  
$e_n, \tilde{e}_n$  
$e_{S, n}$  
$L_\gamma(G)$  
$\hat{\gamma}_i, L_\gamma(G)$  
$(K, \sigma, \beta), (\alpha, \beta; F; \zeta)_n, (\alpha, \beta)_n$  
$UD(n, F)$  
$(K, G, (c_{\sigma, \tau}))$  
$R_1 \sim R_2, R \sim 1, [R], \text{Br}(F)$  
$[R:F]$  
$res_{L/F}, \deg(R), \text{ind}(R)$  
$C_R(A)$  
$\exp(R)$  
$\text{Br}(F)_m$  
$\text{tr}_{\text{red}}, N_{\text{red}}$  
$cor_{E/F}$
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<tr>
<td>$V_D, P_D, \Gamma_D, \bar{D}$</td>
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<td>$e, e(D, F), f, f(D, F)$</td>
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<td>$\mathcal{B}_R$</td>
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<td>$\text{Hom}(C, -) f_#$</td>
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<td>$\text{Hom}(-, C), f^#, M \otimes_R -, - \otimes T M$</td>
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<td>$\text{pd gldim}$</td>
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<td>$K_0(R)$</td>
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<td>$\text{Ch}, \text{Ch}(C)$</td>
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<td>$B, B_n(A), Z, Z_n(A), H_n(A)$</td>
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<td>$(S(A), (S(d))$</td>
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<td>$L_nF, R^nF$</td>
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<td>$\text{Tor}_n, \text{Ext}^n$</td>
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<td>$R'$, $M^*$</td>
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<td>$T(M), \tau, \tau'$</td>
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<td>$(R, R', M, M', \tau, \tau')$</td>
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<td>$R^e$</td>
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<td>$p, J$</td>
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<td>$M^R$</td>
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<td>$\hat{\Gamma}, F[\Gamma]$</td>
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<td>$\Delta: C \to C \otimes C, \epsilon: C \to F$</td>
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<td>$(C, \Delta, \epsilon), \Delta(a) = \sum a_1 \otimes a_2$</td>
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<td>$f \ast g, C^*, S$</td>
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<td>$M^H, M^{\infty H}$</td>
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<td>$A#H$</td>
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Prerequisites

As mentioned in the Introduction, most of Part IV of Volume 2 is self-contained, modulo some basic results on rings and modules. In Chapter 17, we need a few extra general basic results, mostly concerning finitely generated groups, which we list here.

Finitely generated (f.g.) groups.

A fair part of Chapter 17 concerns f.g. groups, introduced briefly in Volume 1, namely on p. 13 and Exercises 0.23–0.27. Often we look for f.g. subgroups of a given f.g. group. The following straightforward facts often come in handy. Recall that a subgroup $H$ has finite index $G$ if $H$ has finitely many cosets in $G$, the number of which is designated as $[G:H]$.

**Remark 00.1.** Any subgroup $H$ of finite index in a f.g. group $G$ is also f.g. (This was stated in Exercise 0.27 of Volume 1, with an extensive hint.) The same proof shows, more precisely, that if $G$ is generated by $t$ elements and $[G:H] = m$, then $H$ is generated by $tm$ elements.

**Lemma 00.2.** For any $n \in \mathbb{N}$, any f.g. group $G$ has finitely many subgroups of index $n$.

**Proof.** We elaborate on Exercise 0.25 of Volume 1. For any subgroup $H$ of index $n$, we have a homomorphism $\psi_H : G \rightarrow S_n$, given by left multiplication on the cosets of $H$. But any element $a$ of $\ker \psi_H$ satisfies $aH = H$, implying $\ker \psi_H \subseteq H$, and thus $H = \psi_H^{-1}(\mathbb{T})$ for some subgroup $\mathbb{T}$ of $S_n$.

Working backwards, since $G$ is f.g., there are only finitely many homomorphisms from $G$ to $S_n$, which has finitely many possible subgroups $\mathbb{T}$. Since any subgroup $H$ of index $n$ can be recovered in this way, we have only finitely many possibilities for $H$. □
Proposition 00.3. If $H$ is a f.g. normal subgroup of $G$, and $K$ is a subgroup of finite index in $H$, then $K$ contains a f.g. normal subgroup of $G$ that has finite index in $H$. (The special case for $H = G$ was given in Exercise 0.25 of Volume 1.)

Proof. For each $g \in G$, $gKg^{-1}$ is a subgroup of $gHg^{-1} = H$ of the same index as $K$; by the lemma, there are only finitely many of these, so, by Exercise 0.24 of Volume 1, $\bigcap_{g \in G} gKg^{-1}$ is a normal subgroup of $G$ having finite index in $H$. □

Groups of fractions.

In the proof of Theorem 17.61 we also need the following easy special case of the construction of Exercise 8.26 of Volume 1:

Definition 00.4. Suppose $(A, +)$ is a torsion-free Abelian group. The group $A_{\mathbb{Q}}$ is defined as follows:

Define an equivalence on $A \times \mathbb{N}^+$ by putting $(a, m) \sim (b, n)$, iff $an = bm$. Writing $\frac{a}{m}$ for the equivalence class $[(a, m)]$, we define $A_{\mathbb{Q}}$ to be the set of equivalence classes, endowed with the operation

$$\frac{a}{m} + \frac{b}{n} = \frac{an + bm}{mn}.$$ 

Remark 00.5. $A_{\mathbb{Q}}$ is a group, and in fact is a $\mathbb{Q}$-module in the natural way, namely

$$\frac{u}{v} \cdot \frac{a}{m} = \frac{ua}{vm}, \quad a \in A, \ u \in \mathbb{Z}, \ m, v \in \mathbb{N}^+.$$ 

There is a group injection $A \to A_{\mathbb{Q}}$ given by $A \mapsto \frac{a}{1}$. Furthermore, any automorphism $\sigma$ of $A$ extends naturally to an automorphism of $A_{\mathbb{Q}}$ via the action $\sigma(\frac{a}{m}) = \frac{\sigma(a)}{m}$.

(The verifications are along the lines of those in the proof of Proposition 12.18 of Volume 1. Alternatively, once we have tensor products from Chapter 18, we could view $A_{\mathbb{Q}}$ as $A \otimes_{\mathbb{Z}} \mathbb{Q}$.)

Jordan decomposition.

The Jordan decomposition of Theorem 2.75 of Volume 1 has an easy but useful application in nonzero characteristic:

Proposition 00.6. Over a field of characteristic $p > 0$, any $n \times n$ matrix $T$ has a power whose radical component is 0.
**Proof.** Write the Jordan decomposition $T = T_s + T_n$, where the semisimple component $T_s$ and the nilpotent component $T_n$ commute. Then, as in Corollary 4.69 of Volume 1,

$$T^p = (T_s + T_n)^p = T_s^p + T_n^p$$

for each $k$, but $T_n^k = 0$ whenever $p^k > n$, so we conclude for such $k$ that $T^p = T_s^p$ is semisimple. □

**Galois theory.**

We also need a fact from Galois theory, which was missed in Volume 1.

**Proposition 00.7.** Suppose $F$ is a finite field extension of $\mathbb{Q}$, and $a \in F$ is integral over $\mathbb{Z}$. If $|\sigma(a)| \leq 1$ for every embedding $\sigma: F \to \mathbb{C}$, then $a$ is a root of unity.

**Proof.** The minimal monic polynomial $f_a \in \mathbb{Z}[\lambda]$ of $a$ over $\mathbb{Z}$ has some degree $n$; its coefficients are sums of products of conjugates of $a$, and so by hypothesis have absolute value $\leq n$. But there are at most $(2n + 1)^n$ possibilities for such a polynomial; moreover, the hypothesis also holds for each power of $a$, which must thus be a root of one of these polynomials. We conclude that there are only finitely many distinct powers of $a$, which means $a$ is a root of unity. □

**The trace bilinear form.**

We need a result about the **trace bilinear form** on the matrix algebra $M_n(F)$ over a field $F$, given by $\langle x, y \rangle = \text{tr}(xy)$. Clearly this form is symmetric and also nondegenerate, for if $x = (a_{ij})$ with $a_{i_0 j_0} \neq 0$, then $\text{tr}(x e_{i_0 j_0}) = a_{i_0 j_0} \neq 0$. The **discriminant** of a base $B = \{b_1, \ldots, b_n\}$ of $M_n(F)$ is defined as the determinant of the $n^2 \times n^2$ matrix $(\text{tr}(b_i b_j))$. In view of Remark 4B.5 of Volume 1, the discriminant of any base $B$ is nonzero (since there exists an orthogonal base with respect to the trace bilinear form).

**Lemma 00.8.** Suppose $\{b_1, \ldots, b_n\}$ is a base of $M_n(F)$ over $F$. Then for any $\alpha_1, \ldots, \alpha_n^2 \in F$, the system of $n^2$ equations $\{\text{tr}(b_i x) = \alpha_i : 1 \leq i \leq n^2\}$ has at most one solution for $x \in M_n(F)$.

**Proof.** Write $x = \sum_{j=1}^{n^2} \gamma_j b_j$. Then $\alpha_i = \sum_{j=1}^{n^2} \gamma_j \text{tr}(b_i b_j)$, $1 \leq i \leq n^2$, can be viewed as $n^2$ equations in the $\gamma_j$; since the discriminant $\det(\text{tr}(b_i b_j))$ is nonzero, one can solve these equations using Cramer’s rule.

To prove uniqueness, suppose there were two matrices $x_1$ and $x_2$ such that $\text{tr}(b_i x_1) = \text{tr}(b_i x_2)$, $1 \leq i \leq n^2$. Then $\text{tr}(b_i (x_1 - x_2)) = 0$ for each $i$, which implies $x_1 - x_2 = 0$ since the trace form is nondegenerate; thus, $x_1 = x_2$. □
List of Major Results

The prefix E denotes that the referred result is an exercise, such as E0.5. Since the exercises do not necessarily follow a chapter immediately, their page numbers may be out of sequence.

Prerequisites.

00.3. Any subgroup of finite index in a f.g. normal subgroup $H$ contains a f.g. normal subgroup of $G$ of finite index in $H$ xxiv

00.6. Any $n \times n$ matrix has a power that is semisimple. xxiv

00.7. If $a \in F$ is integral over $\mathbb{Z}$ and $|\sigma(a)| \leq 1$ for every embedding $\sigma: F \to \mathbb{C}$, then $a$ is a root of unity. xxv

Chapter 13.

13.9. Any ring $W$ having a set of $n \times n$ matrix units has the form $M_n(R)$, where $R = e_{11}We_{11}$. 10

13.14. There is a lattice isomorphism \{Ideals of $R$\} $\to$ \{Ideals of $M_n(R)$\} given by $A \mapsto M_n(A)$. 13

13.18. For any division ring $D$, the ring $M_n(D)$ is a direct sum of $n$ minimal left ideals, and thus has composition length $n$. 14

13.31. Determination of the modules, left ideals, and ideals for a finite direct product of rings. 18

13.40 (Schur’s Lemma). If $M$ is a simple module, then $\text{End}_R M$ is a division ring. 22

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13.42. $M_n(W) \cong (\text{End}_W W^{(n)})^{\text{op}}$ as rings.

13.44. $\text{Hom}(\bigoplus_{i \in I} M_i, \bigoplus_{j \in J} N_j)_W \cong \bigoplus_{i,j} \text{Hom}(M_i, N_j)_W$ as additive groups, for any right $W$-modules $M_i, N_j$.

13.47. $\text{End}_R(S_{1})^{1} \oplus \cdots \oplus S_{t}^{t} \cong \prod_{i=1}^{t} M_{n_{i}}(D_{i})$, for simple pairwise nonisomorphic simple $R$-modules $S_{i}$, where $D_{i} = \text{End} S_{i}$.

13.53. For any division ring $D$, the polynomial ring $D[\lambda]$ satisfies the Euclidean algorithm and is a PLID.

E13.9. Any 1-sum set of orthogonal idempotents $e_{1}, \ldots, e_{n}$, yields the Peirce decomposition $R = \bigoplus_{i=1}^{n} e_{i}Re_{j}$.

E13.24. The power series ring $R[[\lambda]]$ is a domain when $R$ is a domain; $R[[\lambda]]$ is Noetherian when $R$ is Noetherian.

E13A.7. If a ring $W$ contains a left Noetherian subring $R$ and an element $a$ such that $W = R + aR = R + Ra$, then $W$ also is left Noetherian.

E13A.8. Any Ore extension of a division ring is a PLID.

Chapter 14.

14.8. Any submodule of a complemented module is complemented.

14.13. A module $M$ is semisimple iff $M$ is complemented, iff $M$ has no proper large submodules.

14.16. A semisimple module $M$ is Artinian iff $M$ is Noetherian, iff $M$ is a finite direct sum of simple submodules.

14.19. A ring $R$ is semisimple iff $R \cong \prod_{i=1}^{t} M_{n_{i}}(D_{i})$ for suitable division rings $D_{i}$.

14.23. Any module over a semisimple ring is a semisimple module.

14.24 (Wedderburn-Artin). A ring $R$ is simple with a minimal (nonzero) left ideal iff $R \cong M_{n}(D)$ for a division ring $D$.

14.27. Any f.d. semisimple algebra over an algebraically closed field $F$ is isomorphic to a direct product of matrix algebras over $F$.

14.28 (Another formulation of Schur’s Lemma). Suppose, for $F$ an algebraically closed field, $M = F^{(n)}$ is simple as an $R$-module. Then any endomorphism of $M$ is given by scalar multiplication.

E14.8. $\text{soc}(M) = \bigcap \{\text{Large submodules of } M\}$. 

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E14.21. If $R$ is simple and finite-dimensional over an algebraically closed field $F$, and $R$ has an involution $(\ast)$, then $(R, \ast) \cong (M_n(F), J)$, where $J$ is either the transpose or the canonical symplectic involution. 167

Chapter 15.

15.7. If a prime ring $R$ has a minimal nonzero left ideal $L$, then $R$ is primitive and every faithful simple $R$-module is isomorphic to $L$. 47

15.9, 15.10. The Wedderburn-Artin decomposition $R = M_n(D)$ of a simple Artinian ring is unique. Every semisimple ring has finitely many simple nonisomorphic modules. 48

15.18–15.20. Any left Artinian ring $R$ has only finitely many primitive ideals, and each primitive ideal is maximal. Their intersection is the Jacobson radical $J$, which is nilpotent, and $R/J$ is a semisimple ring. Consequently, any prime left Artinian ring is simple Artinian; any semiprime left Artinian ring is semisimple Artinian. 50, 51

15.21 (Hopkins-Levitzki). Any left Artinian ring is also left Noetherian. 52

15.23. If $R$ is left Artinian and $N$ is a nil subset satisfying the condition that for any $a_1, a_2$ in $N$ there is $\nu = \nu(a_1, a_2) \in \mathbb{Z}$ with $a_1a_2 + \nu a_2 a_1 \in N$, then $N$ is nilpotent. 52

15.26 (Wedderburn’s Principal Theorem). If $R$ is a f.d. algebra over an algebraically closed field $F$, then $R = S \oplus J$ where $S$ is a subalgebra of $R$ isomorphic to $R/J$. 54

15A.2 (Jacobson Density Theorem for simple modules). Suppose $M$ is a simple $R$-module, and $D = \text{End}_R M$. For any $n \in \mathbb{N}$, any $D$-independent elements $a_1, \ldots, a_n \in M$, and any elements $b_1, \ldots, b_n$ of $M$, there is $r \in R$ such that $r a_i = b_i$ for $1 \leq i \leq n$. 57

15A.4. If $A$ is a subalgebra of $M_n(F) = \text{End} F^{(n)}$ for $F$ an algebraically closed field, and $F^{(n)}$ is simple as an $A$-module, then $A = M_n(F)$. 58

15A.5 (Amitsur). $\text{Jac}(R[\lambda]) = 0$ whenever $R$ has no nonzero nil ideals. 58

15A.8 (Amitsur). If $R$ is a division algebra over a field $F$ such that $\dim_F R < |F|$, then $R$ is algebraic over $F$. 60

15B.4 (Kolchin). If $S$ is a monoid of unipotent matrices of $M_n(F)$ with $F$ algebraically closed field $F$, then $S$ can be simultaneously triangularized via a suitable change of base. 61
E15.3. A ring $R$ is primitive iff $R$ has a left ideal comaximal with all prime ideals.

E15.6. Any prime ring having a faithful module of finite composition length is primitive.

E15.7. For $W = \text{End}_D M$ and $f \in W$, the left ideal $Wf$ is minimal iff $f$ has rank 1. Also, the set of elements of $W$ having finite rank is an ideal of $W$, which is precisely $\text{soc}(W)$.

E15.21. For any semiprime ring, $\text{soc}(R)$ is also the sum of the minimal right ideals of $R$.

E15.24. $\text{Jac}(R)$ is a quasi-invertible ideal that contains every quasi-invertible left ideal of $R$.

E15.26. $\text{Jac}(R)$ is the intersection of all maximal right ideals of $R$.

E15A.1. For any faithful simple $R$-module $M$ that is infinite-dimensional over $D = \text{End}_R M$, and each $n$, $M_n(D)$ is isomorphic to a homomorphic image of a subring of $R$.

E15A.3. If $W$ is a finite normalizing extension of $R$, then any simple $W$-module is a finite direct sum of simple $R$-modules.

E15A.4. $\text{Jac}(R) \subseteq \text{Jac}(W)$ for any finite normalizing extension $W$ of $R$.

E15A.6. $R \cap \text{Jac}(W) \subseteq \text{Jac}(R)$ whenever the ring $R$ is a direct summand of $W$ as an $R$-module.

E15A.8. For any algebra $W$ over a field, every element of $\text{Jac}(W)$ is either nilpotent or transcendental.

E15A.9 (Amitsur). $\text{Jac}(R)$ is nil whenever $R$ is an algebra over an infinite field $F$ satisfying the condition $\dim_F R < |F|$.

E15B.9. Kolchin’s Problem has an affirmative answer for locally solvable groups and for locally metabelian groups.

E15B.12. (Derakhshan). Kolchin’s Problem has an affirmative answer in characteristic 2.

Chapter 16.

16.17. If $L < R$ and $Rs \cap L = 0$ with $s \in R$ left regular, then the left ideals $L, Ls, Ls^2, \ldots$ are independent.
16.23 (Goldie). A ring $R$ has a semisimple left ring of fractions iff $R$ satisfies the following two properties: (i) $Rs<_{s}R$ for each regular element $s$. (ii) Every large left ideal $L$ of $R$ contains a regular element.

16.24. Any ring $R$ satisfying ACC(ideals) has only finitely many minimal prime ideals, and some finite product of them is 0.

16.26 (Levitzki). Any semiprime ring satisfying ACC on left ideals of the form $\{\ell(r) : r \in R\}$ has no nonzero nil right ideals and no nonzero nil left ideals.

16.29 (Goldie). Any semiprime left Noetherian ring has a semisimple left ring of fractions. Any prime left Noetherian ring $R$ has a simple Artinian left ring of fractions.

16.31. Generalization of Theorem 15.23 to left Noetherian rings.

16.35. Any left Noetherian ring $R$ has IBN.

16.46 (Fitting’s Lemma). If $M$ has finite composition length $n$, then $M = f^n(M) \oplus \ker f^n$ for any map $f: M \to M$; furthermore, $f$ restricts to an isomorphism on $f^n(M)$ and a nilpotent map on $\ker f^n$.

E16.4 (Levitzki). A ring $R$ is semiprime iff $N(R) = 0$.

E16.6. The upper nilradical of $R$ is the intersection of certain prime ideals, and is a nil ideal that contains all the nil ideals of $R$.

E16.8. If $R$ is weakly primitive, then $R$ is a primitive ring.


E16.15 (Goldie’s Second Theorem). A ring $R$ has a semisimple left ring of fractions iff $R$ is a semiprime left Goldie ring.

E16.16 (Goldie’s First Theorem). The ring of fractions of any prime Goldie ring is simple Artinian.

E16.17. $ab = 1$ implies $ba = 1$ in a left Noetherian ring.

E16.25 (Martindale). If $R$ is a prime ring and $a, b \in R$ with $arb = bra$ for all $r \in R$, then $a = cb$ for some $c$ in the extended centroid.

E16.29. (Wedderburn-Krull-Schmidt-Azumaya-Beck). For any finite direct sum of LE-modules, every other decomposition as a direct sum of indecomposables is the same, up to isomorphism and permutation of summands. In particular, this is true for modules of finite composition length.
E16.30. Suppose the ring $R = Re_1 \oplus \cdots \oplus Re_t = Re'_1 \oplus \cdots \oplus Re'_t$ is written in two ways as a direct sum of indecomposable left ideals. Then $t' = t$ and there is some invertible element $u \in R$ and permutation $\pi$ such that $e'_{\pi(i)} = ue_iu^{-1}$ for each $1 \leq i \leq t$.

E16.33. A graded module $M$ is gr-semisimple iff every graded submodule has a graded complement.

E16.34 (Graded Wedderburn-Artin.). Any gr-left Artinian, gr-simple ring has the form $\text{END}(M)_D$, where $M$ is f.g. over a gr-division ring $D$.

E16.36 (Graded First Goldie Theorem – Goodearl and Stafford). If $R$ is graded by an Abelian group $G$ and is gr-prime and left gr-Goldie, then $R$ has a gr-simple left gr-Artinian graded ring of (left) fractions.

E16.40 (Bergman). $\text{Jac}(R)$ is a graded ideal of any $\mathbb{Z}$-graded ring $R$.

E16A.4. The quantized matrix algebra, quantum affine space, and the quantum torus all are Noetherian domains.

Chapter 17.

17.12. Any domain $R$ is either an Ore domain or contains a free algebra on two generators.

17.16 (The Pingpong Lemma). Suppose a group $G$ acts on a set $S$, and $A, B \subseteq G$. If $S$ has disjoint subsets $\Gamma_A$ and $\Gamma_B$ satisfying $a\Gamma_B \subseteq \Gamma_A$, $b\Gamma_A \subseteq \Gamma_B$, and $b\Gamma_B \cap \Gamma_B \neq \emptyset$ for all $a \in A \setminus \{e\}$ and $b \in B \setminus \{e\}$, then $A$ and $B$ interact freely.

17.20. If $0 \rightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \rightarrow M_k \rightarrow 0$ is an exact sequence of f.g. modules over a left Artinian ring, then $\sum_{j=1}^k (-1)^j \ell(M_j) = 0$.

17.25 (König Graph Theorem). Any infinite connected, directed graph has an infinite path.

17.38. The Hilbert series of a commutative affine algebra is rational.

17.49. Any commutative affine algebra has integral Gel’fand-Kirillov dimension, equal both to its Krull dimension and to its transcendence degree. For any algebra with filtration whose associated graded algebra is commutative affine, the Gel’fand-Kirillov dimension is an integer.

17.55 (Bergman Gap Theorem). The Gel’fand-Kirillov dimension cannot be between 1 and 2.
17.60. The growth rate of each nilpotent group is polynomially bounded. 115

17.61 (Milnor-Wolf). Any f.g. virtually solvable group of subexponential growth is virtually nilpotent. 117

17.66. Every f.g. linear group of subexponential growth is of polynomial growth. 120

17A.9 (Nielsen-Schreier). Every subgroup of a free group is free. 124

17B.5 (The Diamond Lemma). A reduction procedure is reduction-unique on $A$ iff for each $r \in A$ and any reductions $\rho, \tau$, the elements $\rho(r)$ and $\tau(r)$ have chains of reductions arriving at the same element. 128

17B.7. The word problem is solvable in any group satisfying Dehn’s algorithm. 130

17B.13 (Bergman). Any set of relations can be expanded to a set of relations for which any given word $h$ becomes reduction-unique. 133

17C.2. The generalized BP has a positive answer for solvable groups. 134

E17.8. The free group on a countably infinite set can be embedded into the free group $G$ on two letters. 181

E17.9. The free group $G$ can be embedded into $GL(2, F)$. 181

E17.13. $D[[M]]$ is a division ring, for any ordered group $M$ and any division ring $D$. 181

E17.22. $\gamma_t/\gamma_{t+1}$ is a free f.g. Abelian group, for every $t$. 182

E17.23 (Magnus-Witt). The free group $G$ is an ordered group. 182

E17.24. $F[[G]]$ is a division ring containing the free algebra $F\{X\}$. 182

E17.31 (Generalized Artin-Tate Lemma). If is an affine algebra is f.g. over a commutative (not necessarily central) subalgebra $C$, then $C$ is affine. 183

E17.32. Any affine algebra that is f.g. over a commutative subalgebra has a rational Hilbert series with respect to a suitable generating set. 183

E17.37. $GK(R/I) \leq GK(R) - 1$ for any $I \triangleleft R$ containing a regular element of $R$. 183

E17.44. Under the hypotheses of Theorem 17.60, the nilpotent group $N$ has polynomial growth of degree $\sum_j jd_j$. 184
E17A.1. The symmetric group \( S_n \) has the Coxeter presentation \( \sigma_i^2 = 1, \)
\((\sigma_i \sigma_{i+1})^3 = 1, \) and \( (\sigma_i \sigma_j)^2 = 1 \) for \(|j - i| > 1\).

E17A.7. Any subgroup of index \( m \) in a free group of rank \( n \) is free of rank \( mn - m + 1 \).

E17A.9. Any group \( G \) is the fundamental group of a complex \( K \) of dimension 2. \( G \) is finitely presented iff \( K \) can be taken finite.

E17B.1. Any set of relations can be expanded to a Gröbner-Shirshov basis.

E17C.1. The Burnside group \( B(m, 3) \) is finite for all \( m \).

E17C.3. The Burnside group \( B(m, 4) \) is finite for all \( m \).

E17C.7, E17C.8. Grigorchuk’s group is infinite but torsion; every element has order a power of 2.

Chapter 18.

18.4. Any balanced map \( \psi: M \times N \to G \) yields a group homomorphism \( \overline{\psi}: M \otimes N \to G \) given by \( \overline{\psi}(a \otimes b) = \psi(a, b) \).

18.5. For any map \( f: M \to M' \) of right \( R \)-modules and map \( g: N \to N' \) of \( R \)-modules, there is a group homomorphism \( f \otimes g: M \otimes_R N \to M' \otimes N' \) given by \( (f \otimes g)(a \otimes b) = f(a) \otimes g(b) \).

18.11. \( (M_1 \oplus \ldots \oplus M_t) \otimes N \cong (M_1 \otimes N) \oplus \cdots \oplus (M_t \otimes N) \).

18.12. Suppose \( M \) is a free right \( R \)-module with base \( B = \{b_i : i \in I\} \), and \( N \) is an \( R \)-module. Then every element of \( M \otimes N \) can be written uniquely in the form \( \sum_{i \in I} b_i \otimes v_i \) for \( v_i \) in \( N \).

18.13. \( C^{(m)} \otimes_C C^{(n)} \cong C^{(mn)} \).

18.15. \( M_1 \otimes_{R_2} (M_2 \otimes_{R_3} M_3) \cong (M_1 \otimes_{R_2} M_2) \otimes_{R_3} M_3 \).

18.16. \( \tau: A \otimes_C B \cong B \otimes_C A \).

18.21. If \( A \) and \( B \) are \( C \)-algebras, then \( A \otimes_C B \) is also a \( C \)-algebra with multiplication \( (a \otimes b)(a' \otimes b') = aa' \otimes bb' \) and \( c(a \otimes b) = ca \otimes b \).

18.25. The following algebra isomorphisms hold for any \( C \)-algebras: \( A \otimes_C C \cong C \otimes_C A \cong A; \ A_1 \otimes A_2 \cong A_2 \otimes A_1; \ A_1 \otimes (A_2 \otimes A_3) \cong (A_1 \otimes A_2) \otimes A_3. \)

18.29'. A finite field extension \( K \supseteq F \) is separable iff the ring \( K \otimes_{F} K \) is semisimple.
18.31. Any splitting field $K$ of an $F$-algebra $R$ contains some subfield $K_0$ f.g. over $F$ such that $K_0$ also splits $R$.

18.33. If $R$ is simple with center a field $F$, and $W$ is an $F$-algebra, then any nonzero ideal $I$ of the tensor product $R \otimes_F W$ contains $1 \otimes w$ for some $w \in W$. In particular, if $W$ is simple, then $R \otimes_F W$ is also simple.

18.36. $M_m(C) \otimes M_n(C) \cong M_{mn}(C)$.

18.41. The tensor product of two integral domains over an algebraically closed field $F$ is an integral domain.

18.42. If $X$ and $Y$ are affine varieties over an algebraically closed field $F$, then $X \times Y$ is an affine variety, with $F[X] \otimes F[Y] \cong F[X \times Y]$.

18.44. $\Phi : \text{Hom}_R(A \otimes_S B, C) \cong \text{Hom}_S(B, \text{Hom}_R(A, C))$.

E18.2. $(\bigoplus_{i \in I} M_i) \otimes N \cong \bigoplus_{i \in I} (M_i \otimes N)$.

E18.7. If $K \rightarrow N \rightarrow P \rightarrow 0$ is an exact sequence of right $R$-modules, then $K \otimes M \rightarrow N \otimes M \rightarrow P \otimes M \rightarrow 0$ is also exact.

E18.12. $C(V, Q)$ has an involution ($\ast$) satisfying $v^* = v$, $\forall v \in V$.

E18.16. For any separable field extension $K$ of $F$, $K \otimes_F K$ has a simple idempotent $e$ with $(a \otimes b) e = (b \otimes a) e$ for all $a, b \in K$.

E18.18 (Wedderburn’s Principal Theorem). Any finite-dimensional algebra $R$ over a perfect field $F$ has a Wedderburn decomposition $R = S \oplus J$ for a suitable semisimple subalgebra $S \cong R/J$ of $R$.

E18.19. The tensor product of two reduced algebras over an algebraically closed field is reduced.

E18.23 (Amitsur). If $R$ is an algebra without nonzero nil ideals over a field $F$, then $\text{Jac}(R \otimes_F F(\lambda)) = 0$.

E18.24. $K \otimes_F \text{Jac}(R) \subseteq \text{Jac}(K \otimes_F R)$ whenever $K \supseteq F$ are fields and $R$ is an algebra over $F$, equality holding if $K/F$ is separable.

Chapter 19.

19.18. For any vector space $V$ over a field $F$, there is a 1:1 correspondence among: group representations $\rho : G \rightarrow \text{GL}(V)$, algebra representations $F[G] \rightarrow \text{End}_F V$, $G$-space structures on $V$, and $F[G]$-module structures on $V$. 


19.22. A group representation $\rho$ of degree $n$ is reducible iff there is a representation $\tau$ equivalent to $\rho$ for which each matrix $\tau(g)$, $g \in G$, has the form (19.4) (for suitable $1 \leq m < n$).

19.26 (Maschke’s Theorem). $F[G]$ is a semisimple ring, for any finite group $G$ whose order is not divisible by $\text{char}(F)$.

19.33. Any finite group $G$ has a splitting field that is finite over $\mathbb{Q}$.

19.36. For any splitting field $F$ of the group $G$, a representation $\rho$ of degree $n$ is irreducible iff $\{\rho(g) : g \in G\}$ spans $M_n(F)$.

19.38. The following are equivalent, for $F$ a splitting field of a finite group $G$: (i) $G$ is Abelian; (ii) The group algebra $F[G]$ is commutative; (iii) $F[G] \cong F \times F \times \cdots \times F$; (iv) Every irreducible representation of $G$ has degree 1.

19.42. $\text{Cent}(C[G])$ is free as a $C$-module.

19.43. The following numbers are equal, for $F$ a splitting field of a finite group $G$: (i) the number of conjugacy classes of $G$; (ii) the number of inequivalent irreducible representations of $G$; (iii) the number of simple components of $F[G]$; (iv) $\dim_F \text{Cent}(F[G])$.

19.48. Any complex irreducible representation of $G$ of degree $n_i$ either is extended from a real irreducible representation or corresponds to a real irreducible representation of degree $2n_i$.

19.61. If $\text{char}(F) = 0$ or $\text{char}(F) > n$, then $I_{\lambda} = \bigoplus_{T_{\lambda} \text{standard}} F[S_n]_{eT_{\lambda}}$.

19.64 (Frame, Robinson, and Thrall). $f_{\lambda} = \frac{n!}{\prod_{k_{ij}} k_{ij}}$.

19A.4. If $\{(a, b) : a \in A, b \in B\}$ is finite for $A, B \triangleleft G$, then the group $(A, B)$ is finite.

19A.9 (Burnside, Schur). In characteristic 0, every linear group of finite exponent is finite, and any f.g. periodic linear group is finite.

19A.12. Every open subgroup of a quasicompact group is closed of finite index.

19A.16. Every continuous f.d. representation of a compact (Hausdorff) group is a finite direct sum of continuous irreducible representations.

19A.19. Any Lie homomorphism $\phi: G \to H$ of Lie groups ($G$ connected) is uniquely determined by its tangent map $d_e\phi$. 
19B.4. In any algebraic group $G$, each open subgroup of $G$ is closed of finite index, each closed subgroup $H$ of $G$ of finite index is open, and $G_e$ is clopen of finite index in $G$. 239

19B.11. If $H \leq G$, then $\overline{H} \leq G$; furthermore, if $H$ contains a nonempty open subset $U$ of $\overline{H}$, then $H$ is closed. 240

19B.19. Every affine algebraic group is linear. 243

19B.21 (The Tits alternative). Every f.g. linear group either is virtually solvable or contains a free subgroup. 244

19B.24 (Breuillard-Gelanter). Any f.g. linear group contains either a free subgroup that is Zariski dense (in the relative topology), or a Zariski open solvable subgroup. 248

E19.6 (Schur’s Lemma, representation-theoretic formulation). For $F$ a splitting field for $G$, $\text{End}_F[G](L_i) \cong F$ and $\text{Hom}_F[G](L_i, L_j) = 0$ for all $i \neq j$, where $L_i$ denotes the module corresponding to $\rho_i$. 355

E19.13. A representation $\rho$ of finite degree $\rho$ is completely reducible whenever its $G$-space has a $G$-invariant Hermitian form. 356

E19.31. $C[G]$ is semiprime, for any group $G$ and any integral domain $C$ of characteristic 0. 358

E19.34 (Herstein; Amitsur). $\text{Jac}(F[G]) = 0$ for any uncountable field $F$ of characteristic 0. 358

E19.42 (Schur’s Double Centralizer Theorem.) Suppose $V$ is any f.d. vector space over a field of characteristic 0. The diagonal action of $\text{GL}(V)$ and the permutation action of $S_n$ on $V^\otimes n = V \otimes \cdots \otimes V$ centralize each other, and provide algebra homomorphisms $\hat{\rho}: F[\text{GL}(V)] \rightarrow \text{End}_F V^\otimes n$ and $\hat{\tau}: F[S_n] \rightarrow \text{End}_F V^\otimes n$. Their respective images are the centralizers of each other in $\text{End}_F V^\otimes n$. 359

E19A.6 (Burnside). Any f.g. periodic linear group is finite. 361

E19A.8 (Schur). Each periodic subgroup of $\text{GL}(n, \mathbb{C})$ consists of unitary matrices with respect to some positive definite Hermitian form. 361

E19A.11 (Jordan). Any unitary subgroup $G \subseteq \text{GL}(n, \mathbb{C})$ has a normal Abelian subgroup of index bounded by $(\sqrt{n} + 1)^{2n^2} - (\sqrt{n} - 1)^{2n^2}$. 362

E19A.16. For any continuous complex representation of degree $n$ of a compact topological group $G$, the vector space $\mathbb{C}^{(n)}$ has a positive definite $G$-invariant Hermitian form. 362
E19A.18. Any continuous f.d. representation of $G$ having a positive
definite $G$-invariant Hermitian form is completely reducible

E19A.30 (Artin’s combing procedure). The kernel of the map $P_n \rightarrow P_{n-1}$
obtained by cutting the $n$-th strand is the free group of rank $n-1$.

E19A.34. The braid group $B_n$ satisfies $B'_n = (B'_n, B_n)$.

E19B.7. The Tits alternative also over fields of any characteristic.

E19B.14. The commutator group of two closed subgroups of an algebraic
\[ G \] group $G$ is closed. In particular, all the derived subgroups of $G$ are closed,
\[ G \] and all subgroups in its upper central series are closed.

E19B.16. For $F$ algebraically closed, any connected solvable algebraic
\[ G \] subgroup $G$ of $GL(n, F)$ is conjugate to a subgroup of $T(n, F)$.

Chapter 20.

20.5. The characters $\chi_1, \ldots, \chi_t$ comprise an orthonormal base of $\mathcal{R}$ with
\[ G \] respect to the Schur inner product.

20.10. $\sum_{g \in G} \chi_i(ga) \overline{\chi_j(g)} = \frac{\delta_{ij}|G|^{|\chi_i(a)|}}{n_i}$ for each $a \in G$.

20.14 (Schur I). $\delta_{ik}|G| = \sum_{j=1}^t m_j \chi_{ij} \overline{\chi_{kj}}$.

20.15 (Schur II). $\sum_{i=1}^t \chi_{ij} \overline{\chi_{ik}} = \delta_{jk} \frac{|G|}{m_k}$.

20.18 (Frobenius). $n_i$ divides $|G|$ for each $i$.

20.20. If $\text{gcd}(n_i, m_j) = 1$, then either $\chi_{ij} = 0$ or $|\chi_{ij}| = n_i$.

20.22. In a finite nonabelian simple group, the size of a conjugacy class
cannot be a power (other than 1) of a prime number.

20.24 (Burnside). Every group of order $p^u q^v$ ($p, q$ prime) is solvable.

20.32. The character table of $G \times H$ is the tensor product of the character
tables of $G$ and of $H$.

20.42 (Frobenius Reciprocity Theorem). For $F \subseteq \mathbb{C}$ a splitting field of
\[ G \] a finite group $G$, if $\sigma$ is an irreducible representation of a subgroup $H$ and
\[ G \] $\rho$ is an irreducible representation of $G$, then the multiplicity of $\rho$ in $\sigma^G$
is the same as the multiplicity of $\sigma$ in $\rho^H$.

20.43. For $H < K < G$ and a representation $\rho$ of $H$, the representations
\[ G \] $(\rho^K)^G$ and $\rho^G$ are equivalent, $(\rho_1 \oplus \rho_2)^G$ and $\rho_1^G \oplus \rho_2^G$, are equivalent, and
\[ G \] $\rho^G \otimes \sigma$ and $(\rho \otimes \sigma_H)^G$ are equivalent for any representation $\sigma$ of $G$. 
20.44 (Artin). Every complex character of a group is a linear combination (over $\mathbb{Q}$) of complex characters induced from cyclic subgroups.

E20.20. $n_i$ divides $[G:Z_i]$ for each $i$.


E20.27. For any representation $\rho$ of finite degree of a subgroup $H \subseteq G$, the contragredient $(\rho^G)^*$ of the induced representation is equivalent to the induced representation $(\rho^*)^G$.

Chapter 21.

21.21. For $F$ algebraically closed, if $L$ is a Lie subalgebra of $\mathfrak{gl}(n, F)$ and $a = s + n$ is the Jordan decomposition of $a \in L$, then $\text{ad}_a = \text{ad}_s + \text{ad}_n$ is the Jordan decomposition of $\text{ad}_a$.

21.27. If $L$ is a Lie subalgebra of $R^-$ and $\text{ad}_a$ is nilpotent for every $a \in L$, then $\text{ad} L$ is nilpotent under the multiplication of $R$, and $L$ is a nilpotent Lie algebra.

21.29 (Engel). Any Lie algebra $L \subseteq \mathfrak{gl}(n, F)$ of nilpotent transformations becomes a Lie subalgebra of the algebra of strictly upper triangular matrices under a suitable choice of base.

21.32 (Lie). If a Lie subalgebra $L$ of $\mathfrak{gl}(n, F)$ acts solvably on $F^{(n)}$, with $F$ an algebraically closed field, then $L$ acts in simultaneous upper triangular form with respect to a suitable base of $F^{(n)}$.

21.38. If $L \subseteq \mathfrak{gl}(n, F)$ in characteristic 0 such that $\text{tr}(aL') = 0$ for all $a \in L$, then $L'$ is a nilpotent Lie algebra.


21.47 (Cartan’s second criterion). A f.d. Lie algebra $L$ of characteristic 0 is semisimple iff its Killing form is nondegenerate.

21.51. Any f.d. semisimple Lie algebra $L$ of characteristic 0 is a direct sum $\bigoplus S_i$ of simple nonabelian Lie subalgebras $S_i$, with each $S_i \triangleleft L$, and any Lie ideal of $L$ is a direct sum of some of the $S_i$.

21.53. The trace bilinear form of any representation $\rho$ of a f.d. semisimple Lie algebra is nondegenerate.

21.54 (Zassenhaus). Every derivation of a f.d. semisimple Lie algebra $L$ of characteristic 0 is inner.
21.57. The Casimir element satisfies \( \text{tr}(c_\rho) = n \) and \([\rho(L), c_\rho] = 0\). 292

21.58 (Weyl). Any f.d. representation of a f.d. semisimple Lie algebra \( L \) (of characteristic 0) is completely reducible. 292

21.61. For any given nilpotent Lie subalgebra \( N \) of a f.d. Lie algebra \( L \), there exists a unique root space decomposition \( L = \bigoplus \mathbb{L}_a \). 295

21.64. \( L_b \perp L_a \) for any roots \( a \neq -b \). 296

21.71, 21.72. Any f.d. semisimple Lie algebra over an algebraically closed field of characteristic 0 has a Cartan subalgebra \( \mathfrak{h} \), which is its own nullspace under the corresponding root space decomposition. The restriction of the Killing form to \( \mathfrak{h} \) is nondegenerate. \( \mathfrak{h} \) is Abelian, and \( \text{ad}_h \) is semisimple for all \( h \in \mathfrak{h} \). 298

21.79. For any root \( a \), \( \dim L_a = \dim L_{-a} = 1 \), and \( k a \) is not a root whenever \( 1 < |k| \in \mathbb{N} \). 301

21.80. \( \langle h_1, h_2 \rangle = \sum_{a \neq 0} a(h_1)a(h_2) \), \( \forall h_1, h_2 \in \mathfrak{h} \). 301

21.84. Any simple \( \hat{L}_a \)-module \( V \) has an eigenspace decomposition \( V = V_m \oplus V_{m-2} \oplus \cdots \oplus V_{-(m-2)} \oplus V_m \), where each component \( V_{m-2j} = Fv_j \) is a one-dimensional eigenspace of \( h_a \) with eigenvalue \( m - 2j \). In particular, \( V \) is determined up to isomorphism by its dimension \( m + 1 \). 303

21.88. \([L_a L_b] = L_{b+a} \) whenever \( a, b, \) and \( b + a \) are roots. 305

21.91. \( \langle a, a \rangle > 0 \) and \( \langle a, b \rangle \in \mathbb{Q} \) for all nonzero roots \( a, b \). The bilinear form given by Equation (21.18) restricts to a positive form on \( \mathfrak{h}^* \), the \( \mathbb{Q} \)-subspace of \( \mathfrak{h}^* \) spanned by the roots, and \( \mathfrak{h}^* = \mathfrak{h}^*_0 \otimes _\mathbb{Q} F \). 306

21.96. \( \langle a, b \rangle \leq 0 \) for all \( a \neq b \in P \). 308

21.97. The set of simple roots is a base of the vector space \( V \) and is uniquely determined by the given order on \( V \). 308

21.102. The Cartan numbers \( m_{ij} \) satisfy \( m_{ij}m_{ji} < 4 \). 310

21.103. The Cartan numbers are integers. 311

21.108. Suppose \( S = \{a_1, \ldots, a_n\} \) is a simple root system for the semisimple Lie algebra \( L \). Take \( e_i \in L_{a_i}, e'_i \in L_{-a_i} \), and \( h_i = [e_i f_i] \). Writing any positive root \( a = a_{i_1} + \cdots + a_{i_t} \), let \( x_a = [e_{i_1} e_{i_2} \cdots e_{i_t}] \) and \( y_a = [f_{i_1} f_{i_2} \cdots f_{i_t}] \). Then \( \{h_1, \ldots, h_n\} \) together with the \( x_a \) and \( y_a \) comprise a base of \( L \). 313
21.110. The Lie multiplication table of $L$ (with respect to the base in Theorem 21.108) has rational coefficients.

21.111. The split f.d. semisimple Lie algebra $L$ is simple iff its simple root system is indecomposable.

21.115, 21.116. Any indecomposable generalized Cartan matrix $A$ is of finite, affine, or indefinite type. The symmetric bilinear defined by $A$ is positive definite iff $A$ has finite type, and is positive semidefinite (of corank 1) iff $A$ has affine type.

21B.18, 21B.19. Suppose the composition algebra $(A, \ast)$ is the $\nu$-double of $(A, \ast)$. If $A$ is associative, then $A$ is alternative. If $A$ is associative, then $A$ must be commutative.

21B.22. (Herstein). If $R$ is a simple associative algebra, with $\frac{1}{2} \in R$, then $R^+$ is simple as a Jordan algebra.

21C.5. (PBW Theorem). The map $\nu_L:L \to U(L)^-$ is 1:1.

E21.10. In characteristic $\neq 2$, the classical Lie algebra $B_n$ is simple for each $n \geq 1$, and $C_n$ and $D_n$ are simple Lie algebras for all $n > 2$.

E21.27 (Herstein). For any associative simple ring $R$ of characteristic $\neq 2$, the only proper Lie ideals of $R'$ are central.

E21.28 (Herstein). If $T$ is an additive subgroup of a simple ring $R$ of characteristic $\neq 2$ such that $[T, R'] \subset T$, then either $T \supset R'$ or $T \subset Z$.

E21.41. The radical of a Lie algebra is contained in the radical of the trace bilinear form with respect to any representation.

E21.42. The Casimir element of an irreducible Lie representation is always invertible.

E21.44 (Whitehead’s First Lemma). For any f.d. Lie module $V$ and linear map $f:L \to V$ satisfying $f([ab]) = af(b) - bf(a)$, $\forall a,b \in L$, there is $v \in V$ such that $f(a) = av$, $\forall a \in L$.

E21.47 (Whitehead’s Second Lemma). For any f.d. semisimple Lie algebra $L$ of characteristic 0 and f.d. Lie module $V$ with $f:L \times L \to V$ satisfying $f(a, a) = 0$ and $\sum_{i=1}^{3} f(a_i, [a_{i+1}, a_{i+2}]) + a_i f(a_{i+1}, a_{i+2}) = 0$, subscripts modulo 3, there is a map $g:L \to V$ with $f(a_1, a_2) = a_1 g(a_2) - a_2 g(a_1) - g([a_1 a_2])$.

E21.48 (Levi’s Theorem). Any f.d. Lie algebra $L$ of characteristic 0 can be decomposed as vector spaces $L = S \oplus I$, where $I = \text{rad} L$ and $S \cong L/I$ is a semisimple Lie subalgebra.
E21.50. $L' \cap \text{rad}(L)$ is Lie nilpotent, for any f.d. Lie algebra $L$ of characteristic 0. 377

E21.60. $(\mathbf{a}, \mathbf{a}) = \sum b \langle \mathbf{a}, \mathbf{b} \rangle^2$ for any root $\mathbf{a}$. 378

E21.63. The formulas $[e_{j_1} e_{j_2} \cdots e_{j_\ell} h_i] = -\sum_{u=1}^\ell m_{ij_u} [e_{j_1} e_{j_2} \cdots e_{j_u}]$ and $[f_{j_1} f_{j_2} \cdots f_{j_\ell} h_i] = \sum_{u=1}^\ell m_{ij_u} [f_{j_1} f_{j_2} \cdots f_{j_u}]$ hold in Theorem 21.108. 378

E21.67. Every root system of a simple Lie algebra $L$ has a unique maximal root. 379

E21.70. The Weyl group acts transitively on simple root systems. 379


E21.75. Equivalent conditions for an indecomposable, symmetric generalized Cartan matrix to have finite type. 380


E21.79 (Farkas). For $\mathbf{a}_i = (\alpha_{i1}, \ldots, \alpha_{i\ell}), 1 \leq i \leq k$, the system $\sum_j \alpha_{ij} \lambda_j > 0$ of linear inequalities for $1 \leq i \leq k$ has a simultaneous solution over $\mathbb{R}$ iff every non-negative, nontrivial, linear combination of the $\mathbf{a}_i$ is nonzero. 381

E21.80 (The Fundamental Theorem of Game Theory). If there does not exist $x > 0$ in $\mathbb{R}^{(\ell)}$ with $Ax < 0$, then there exists $w \geq 0$ (written as a row) in $\mathbb{R}^{(k)}$ with $wA \geq 0$. 381

E21.81. The generalized Cartan matrix $A^t$ has the same type as $A$. 381

E21.90, E21.91 (Farkas-Letzter). For any prime ring $R$ with a Poisson bracket, there exists $c$ in the extended centroid of $R$ such that $[a, b] = c\{a, b\}$ for every $a, b \in R$. 383

E21A.3. The Lie product in $T(G)e$ corresponds to the natural Lie product of derivations in Lie$(G)$. 383

E21A.4. $d\varphi: T(G)e \rightarrow T(H)e$ preserves the Lie product. 383

E21A.6. Description of the classical simple Lie algebras as the Lie algebras of the algebraic groups SL, O, and Sp. 384

E21B.3. The base field $K \supset F$ of any algebra can be cut down to a field extension of finite transcendence degree over $F$. 385
E21B.9 (Moufang). Every alternative algebra satisfies the three identities $a(b(ac)) = (aba)c$, $c(a(ba)) = c(aba)$, and $(ab)(ca) = a(bc)a$. 386

E21B.11 (E. Artin). Any alternative algebra generated by two elements is associative. 386

E21B.19. Any composition $F$-algebra must be either $F$ itself, the direct product of two copies of $F$ (with the exchange involution), a quadratic field extension of $F$, a generalized quaternion algebra, or a generalized octonion algebra. 387

E21B.20 (Hurwitz). If $C$ satisfies an identity $\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} z_i^2$, where $z_i$ are forms of degree 2 in the $x_i$ and $y_j$, then $n = 1, 2, 4, \text{or } 8$. 387

E21B.22 (Zorn). Every f.d. simple nonassociative, alternative algebra is a generalized octonion algebra. 388

E21B.26. The Peirce decomposition of an alternative algebra in terms of pairwise orthogonal idempotents. 388

E21B.29. Any simple alternative algebra $A$ containing three pairwise orthogonal idempotents $e_1, e_2$, and $e_3$ is associative. 388

E21B.37 (Glennie). Any special Jordan algebra satisfies the Glennie identity. 389

E21B.39. (Herstein). $S(R, \ast)$ is Jordan simple for any simple associative algebra with involution of characteristic $\neq 2$. 390

E21C.4. $U(L)$ is an Ore domain, for any Lie algebra $L$ of subexponential growth. 391

E21C.13 (Ado). Any f.d. Lie algebra of characteristic $0$ is linear. 392

E21C.17. $U_q(sl(2, F))$ is a skew polynomial ring. 393

E21C.21. $U_q(L)$ is a Noetherian domain, for any f.d. semisimple Lie algebra $L$ of characteristic $0$. 394

Chapter 22.

22.11. Any connected Dynkin diagram is either $A_n$, $B_n = C_n$, $D_n$, $E_6$, $E_7$, $E_8$, $F_4$, or $G_2$ of Example 22.2. 342

22.13. If any single vertex of the extended Dynkin diagram of a simple affine Lie algebra is erased, the remaining subdiagram is a disjoint union of Dynkin diagrams (of finite type). 345
22.22. For $i \neq j$, the Coxeter bilinear form restricts to a positive semi-
definite form on the two-dimensional subspace $Fe_i + Fe_j$, which is positive
definite iff $\circ(\sigma_i \sigma_j) < \infty$. 348

22.25. The only abstract Coxeter graphs whose quadratic forms are
positive definite are $A_n, D_n, E_6, E_7,$ and $E_8$. 350

22.28. (Bernstein, Gel’fand, and Ponomarev). If an abstract Coxeter
graph $(\Gamma; \nu)$ has only finitely many nonisomorphic indecomposable rep-
resentations, then its quadratic form is positive definite. 352

E22.1–E22.4. Construction of the classical Lie algebras from their Dynkin
diagrams. 394

E22.10. For any generalized Cartan matrix $A$ of affine type, any proper
subdiagram of its Dynkin diagram is the disjoint union of Dynkin diagrams
of simple f.d. Lie algebras. 396

E22.19. Any finite reflection group is Coxeter. 397

E22.20. Any two positive systems $\Phi_1$ and $\Phi_2$ are conjugate under some
element of the Weyl group. 398

E22.23. Each $m_{i,j} \in \{2, 3, 4, 6\}$ for any crystallographic group. 398

E22.26. The bilinear form of any finite Coxeter group $W$ is positive
definite. 398

E22.36. Every finite Coxeter group is a reflection group. 400

Chapter 23.

23.11. Any $t$-alternating polynomial $f$ is an identity for every algebra
spanned by fewer than $t$ elements over its center. 410

23.26 (Razmyslov). There is a 1:1 correspondence between multilinear
central polynomials of $M_n(F)$ and multilinear 1-weak identities that are not
identities. 415

23.31 (Kaplansky). Any primitive ring $R$ satisfying a PI of degree $d$ is
simple of dimension $n^2$ over its center, for some $n \leq \left\lfloor \frac{d}{2} \right\rfloor$. 418

23.32. A semiprime PI-ring $R$ has no nonzero left or right nil ideals. 418

23.33. Any semiprime PI-ring $R$ has some PI-class $n$, and every ideal $A$
intersects the center nontrivially. 419

23.34 (Posner et al). The ring of central fractions of a prime PI-ring $R$
is simple and f.d. over the field of fractions of Cent($R$). 419
23.35. Extension of Theorem 15.23 to PI-rings.

23.39. Suppose $R$ has PI-class $n$ and center $C$, and $1 \in h_n(R)$. Then $R$ is a free $C$-module of rank $n$; also, there is a natural 1:1 correspondence between \{ideals of $R$\} and \{ideals of $C$\}.

23.48. If $R$ is an algebra over an infinite field $F$, and $H$ is any commutative $F$-algebra, then $R$ is PI-equivalent to $R \otimes_F H$.

23.51. The algebra of generic matrices is the relatively free PI-algebra with respect to $\mathcal{I} = \mathcal{M}_{n,C}$.

23.57. Suppose $R$ satisfies a PI of degree $d$, and $1 + \frac{1}{v} \leq \frac{2}{e(d-1)^2}$, where $e = 2.71828 \cdots$. Then any multilinear polynomial of a Young tableau whose shape contains a $u \times v$ rectangle is an identity of $R$.

23A.3 (Shirshov’s Dichotomy Lemma). For any $\ell, d, k$, there is $\beta \in \mathbb{N}$ such that any word $w$ of length $\geq \beta$ in $\ell$ letters is either $d$-decomposable or contains a repeating subword of the form $u^k$ with $1 \leq |u| \leq d$.

23A.5. Any hyperword $h$ is either $d$-decomposable or has the form $vu^\infty$ for some initial subword $v$ and some subword $u$ with $|u| < d$.

23A.6 (Shirshov’s First Theorem). If $R = C \{r_1, \ldots, r_\ell\}$ satisfies a PI, and each word in the $r_i$ of length $\leq d$ is integral over $C$, then $R$ is f.g. as a $C$-module.

23A.7. If $R$ is affine without 1 and satisfies a PI of degree $d$, and if each word in the generators of length $\leq d$ is nilpotent, then $R$ is nilpotent.

23A.10. Any prime PI-algebra and its characteristic closure have a common nonzero ideal.

23A.11 (Kemer). Any affine PI-algebra over a field $F$ of characteristic 0 is PI-equivalent to a finite-dimensional algebra.

23A.19. For any PI algebra $R$, the following assertions are equivalent for any multilinear polynomial $f$ of degree $n$: $f \in \text{id}(R)$; $f_I^* \in \text{id}_2(R \otimes G)$ for some subset $I \subseteq \{1, \ldots, n\}$; $f_I^* \in \text{id}_2(R \otimes G)$ for every subset of $\{1, \ldots, n\}$.

23A.22 (Kemer). Let $R$ be a PI-superalgebra, and $f = f(x_1, \ldots, x_n) = \sum_{\pi \in S_n} \alpha_{\pi} x_{\pi 1} \cdots x_{\pi n}$. Then $f \in \text{id}(G(R))$ iff $f_I^* \in \text{id}_2(R)$ for every subset $I \subseteq \{1, \ldots, n\}$.

23A.23 (Kemer). There is a 1:1 correspondence from \{varieties of superalgebras\} to \{varieties of algebras\} given by $R \mapsto G(R)$. 
23B.5 (Kostrikin-Zelmanov). Over a field of characteristic $p$, any f.g. Lie algebra satisfying the Engel identity $e_{p-1}$ is Lie nilpotent.

442

23B.6 (Zelmanov). If a f.g. restricted Lie algebra $L$ over a field of characteristic $p$ satisfies the Engel identity $e_n$ and all of its partial linearizations, then $L$ is Lie nilpotent.

442

23B.13. The Lie algebra $L$ of a nilpotent group $G$ is indeed a Lie algebra and is $\mathbb{N}$-graded in the sense that $[L_iL_j] \subseteq L_{i+j}$. $L$ is Lie nilpotent of the same index $t$ as the nilpotence class of the group $G$.

444

23B.16 (Kostrikin and Zelmanov). Any sandwich algebra is Lie nilpotent.

446

E23.4. Any algebra that is f.g. as a module over a commutative affine subalgebra is representable.

563

E23.6. The Jacobson radical of a representable affine algebra is nilpotent.

564

E23.16. Every identity of an algebra over a field of characteristic 0 is a consequence of its multilinearizations.

565

E23.17. Over an infinite field, every identity is a sum of completely homogeneous identities.

565

E23.22 (Amitsur-Levitzki). The standard polynomial $s_{2n}$ is an identity of $M_n(C)$ for any commutative ring $C$.

566

E23.24. Every PI-algebra has IBN.

566

E23.26 (Bell). Every prime affine PI-algebra has a rational Hilbert series.

566

E23.30 (Amitsur). Any PI-algebra $R$ satisfies an identity $s_d^k$.

566

E23.32. If algebras $R_1$ and $R_2$ are PI-equivalent, then so are $M_n(R_1)$ and $M_n(R_2)$.

567

E23.36 (Regev). In characteristic 0, the $T$-ideal $\text{id}(G)$ is generated by the Grassmann identity.

567

E23.40 (Regev). $M_n(G(p))$ satisfies the identity $s_{2n}^{p^2p+1}.

568

E23.42 (Kemer). In any $F$-algebra, a suitable finite product of $T$-prime $T$-ideals is 0. Any $T$-ideal has only finitely many $T$-prime $T$-ideals minimal over it.

568
E23B.1. Any simple alternative, nonassociative algebra satisfies the central polynomial $[x, y]^2$.

E23B.4. Any Lie algebra of characteristic 3 satisfying the Engel ad-

E23B.6. For any nilpotent $p$-group $G$ of exponent $n=p^k$, the Lie algebra $L_1(G)$ satisfies the multilinearized $n$-Engel identity $\tilde{e}_n$ and some weak Engel condition $e_{S,2n}$.

E23B.14 (Key step in proving Theorem 23B.16). An enveloping algebra $R$ of a Lie algebra $L$ is nilpotent whenever $R$ is generated by a finite set of 1-thick sandwiches.

E23B.17 (Zelmanov). If a f.g. restricted Lie algebra $L$ satisfies various Engel-type conditions, then its associative enveloping algebra $R$ (without 1) is nilpotent.

Chapter 24.

24.10. If $R_1$ and $R_2$ are csa’s, then $R_1 \otimes_F R_2$ is also a csa.

24.14. If $R$ is a csa, then $\Phi: R \otimes_F R^{\text{op}} \rightarrow \text{End}_F R$ is an isomorphism.

24.15. The Brauer group $\text{Br}(F)$ is a group, where $[R]^{-1} = [R]^{\text{op}}$.

24.23, 24.24. $\text{End}_K R \cong C_R(K) \otimes_F R^{\text{op}}$ as $K$-algebras, for any $F$-subfield $K$ of $R$. $C_R(K)$ is a $K$-csa and $[C_R(K):F] = [R:K]$.

24.25. $R \otimes_K K \sim C_R(K)$ in $\text{Br}(K)$.

24.32 (Double Centralizer Theorem). $C_R(K) \cong A \otimes_K C_R(A)$ and $[A:F][C_R(A):F] = n^2$, for any simple $F$-subalgebra $A$ of a csa $R$, where $K = \text{Cent}(A)$.

24.34 (Index Reduction Theorem). The index reduction factor divides the g.c.d. of $\text{ind}(R)$ and $m = [L:F]$.

24.40 (Skolem-Noether Theorem). Suppose $A_1$ and $A_2$ are isomorphic simple subalgebras of a csa $R$. Any $F$-algebra isomorphism $\varphi: A_1 \rightarrow A_2$ is given by conjugation by some $u \in R^\times$.

24.42 (Wedderburn). Every finite division ring is a field.

24.44. A csa $R$ of degree $n$ over an infinite field $F$ is split iff $R$ contains an element of degree $n$ whose minimal polynomial has a linear factor.

24.48'. $(K, \sigma, \beta_1) \otimes (K, \sigma, \beta_2) \sim (K, \sigma, \beta_1 \beta_2)$.
24.50. Any $F$-csa $R$ is PI-equivalent to $M_n(F)$ for $n = \deg R$.  465

24.51 (Koethe-Noether-Jacobson). Any separable subfield $L$ of a cda $D$ is contained in a separable maximal subfield of $D$.  465

24.52. Every csa is similar to a crossed product.  466

24.54. $UD(n, F)$ is a division algebra of degree $n$ (over its center) for every $n$ and every field $F$ of characteristic prime to $n$.  467

24.57. If $D$ is a cda of degree $p^aq$ with $p$ prime, $p \nmid q$, then there is a field extension $L$ of $F$ with $p \nmid [L : F]$, as well as a splitting field $L_u \supseteq L$ of $D$ together with a sequence of subfields $L_0 = L \subset L_1 \subset L_2 \subset \cdots \subset L_u$ for which $\text{ind}(D \otimes_F L_i) = p^{u-i}$ for each $0 \leq i \leq u$, and each $L_i/L_{i-1}$ is cyclic Galois of dimension $p$.  468

24.62. $\exp(R)$ divides $\text{ind}(R)$. If a prime number $p$ divides $\text{ind}(R)$, then $p$ divides $\exp(R)$.  469

24.66. Any cda $D$ is isomorphic to the tensor product of cda’s of prime power index.  470

24.68 (Wedderburn). Suppose $D$ is a cda. If $a \in D$ is a root of a monic irreducible polynomial $f \in F[\lambda]$ of degree $n$, then $f = (\lambda - a_n) \cdots (\lambda - a_1)$ in $D[\lambda]$, where each $a_i$ is a conjugate of $a$.  472

24.73, 24.74. For any Galois extension $E$ of $F$, $\text{cor}_{E/F}$ induces a homomorphism of Brauer groups, and $\text{cor}_{E/F} \text{res}_{E/F} R \cong R \otimes [E:F]$.  475

24.82 (Cohn-Wadsworth). A cda $D$ has a valuation extending a given valuation $v$ on $F$, iff $v$ extends uniquely to a valuation of each maximal subfield of $D$.  480

24.85 (Hasse). Any cda $D$ of degree $n$ over a local field is a cyclic algebra, having a maximal subfield $K$ isomorphic to the unramified extension of $F$ of dimension $n$.  481

E24.1 (Frobenius). The only $\mathbb{R}$-cda other than $R$ is $\mathbb{H}$.  572

E24.8 (Wedderburn’s criterion). A cyclic algebra $(K, \sigma, \beta)$ of degree $n$ has exponent $n$, if $\beta^j$ is not a norm from $K$ for all $1 \leq j < n$.  573

E24.25. $(K, G, (c_{\sigma, \tau})) \otimes (K, G, (d_{\sigma, \tau})) \sim (K, G, (c_{\sigma, \tau} d_{\sigma, \tau})).$  575

E24.31. Any $p$-algebra is split by a purely inseparable, finite-dimensional field extension.  575
E24.32. If UD(n, F) is a crossed product with respect to a group G, then every F-csa of degree n is a crossed product with respect to G. 576

E24.38. Division algebras of all degrees exist in any characteristic. 577

E24.42. When deg D = 3, any element of reduced norm 1 is a multiplicative commutator. 577

E24.43. When deg D = 3 and char(F) \neq 3, any element of reduced trace 0 is an additive commutator. 577

E24.48 (The Projection Formula). cor_{L/F}(a, b; L) \sim (a, N_{L/F}(b)) when a \in F. 578

E24.49 (Rosset). Any cda D of degree p is similar to the corestriction of a symbol algebra. 578

E24.51. Br(F) is divisible whenever F has enough m-roots of 1. 578

E24.54. e(D/F)f(D/F) \leq [D:F], equality holding when the valuation is discrete and the field F is complete. 579

E24.58. D = (K, \sigma, \pi^n) in Theorem 24.85. 579

E24A.7. (Plücker). The Brauer-Severi variety is a projective variety. 580

E24A.8. A geometric criterion for an n-dimensional subspace of a csa of degree n to be a left ideal. 580

Chapter 25.

25.10. Equivalent conditions for an R-module to be projective. 494

25.11. A direct sum \textstyle \bigoplus P_i of modules is projective iff each of the P_i is projective. 494

25.12'. A ring R is semisimple iff every short exact sequence of R-modules splits, iff every R-module is projective. 495

25.13 (Dual Basis Lemma). An R-module P = \textstyle \sum Ra_i is projective iff there are R-module maps h_i: P \to R satisfying a = \sum_{i \in I} h_i(a)a_i, \forall a \in P, where, for each a, h_i(a) = 0 for almost all i. 495

25.24. If P and Q are modules over a commutative ring C such that \textstyle P \otimes Q \cong C^{(n)}, then P is projective. 501
25.38 (The Snake Lemma). Any commutative diagram
\[
\begin{array}{c}
A'' \xrightarrow{f_1} A_1 \xrightarrow{g_1} A'_1 \xrightarrow{} 0 \\
\downarrow d'' \quad \downarrow d \quad \downarrow d' \\
0 \xrightarrow{} A''_2 \xrightarrow{f_2} A_2 \xrightarrow{g_2} A'_2 \\
\end{array}
\]
gives rise to an exact sequence \( \ker d'' \rightarrow \ker d \rightarrow \ker d' \rightarrow \text{coker } d'' \rightarrow \text{coker } d \rightarrow \text{coker } d' \).

25.44. For any exact sequence \(0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0\) of modules and respective projective resolutions \((P', d')\) and \((P'', d'')\) of \(M'\) and \(M''\), there exists a projective resolution \((P, d)\) of \(M\), such that \(P_n = P'_n \oplus P''_n\) for each \(n\), and the three projective resolutions form a commutative diagram.

25.45. Any short exact sequence \(0 \rightarrow (A'', d'') \xrightarrow{f} (A, d) \xrightarrow{g} (A', d') \rightarrow 0\) of complexes gives rise to a long exact sequence of the homology groups
\[
\cdots \rightarrow H_{n+1}(A'') \xrightarrow{f_*} H_{n+1}(A) \xrightarrow{g_*} H_{n+1}(A') \xrightarrow{\partial_*} H_n(A'') \xrightarrow{f_*} H_n(A) \rightarrow \cdots
\]
where \((\partial_*)_{n+1}: H_{n+1}(A') \rightarrow H_n(A'')\) is obtained via the Snake Lemma.

25.50, 25.51. Given a map \(f: M \rightarrow N\) of modules, a resolution \(A\) of \(N\), and a projective resolution \(P\) of \(M\), one can lift \(f\) to a chain map \(f: P \rightarrow A\) that is unique up to homotopy equivalence. Consequently, any two projective resolutions of a module \(M\) are homotopy equivalent.

25.54. A right exact covariant functor \(F\) is exact iff \(L_1 F = 0\), in which case \(L_n F = 0\) for all \(n\).

25.58. The direct sum \(\bigoplus M_i\) of right modules is flat iff each \(M_i\) is flat.

25.59. Every projective module \(P\) is flat.

25.67 (Shapiro’s Lemma). \(H_n(G, M^L) \cong H_n(L, M)\) for each \(L\)-module \(M\) and all \(n\); \(H^n(G, \text{Coind}^L(G)(M)) \cong H^n(L, M)\) for all \(n\).

25A.8. An \(R\)-module \(M\) is a generator in \(\text{R-Mod}\) iff \(T(M) = R\).

25A.14. If \(R\) and \(R'\) are Morita equivalent rings, then there is an \(R\)-progenerator \(P\) such that \(R' \cong (\text{End}_R P)\text{ op}\).

25A.19 (Morita’s Theorem). Two rings \(R, R'\) are Morita equivalent iff there is an \(R\)-progenerator \(M\) such that \(R' \cong (\text{End}_R M)\text{ op}\); in this case the categorical equivalence \(\text{R-Mod} \rightarrow R'\text{-Mod}\) is given by \(M^* \otimes_R -\).
25A.19’. Notation as in Morita’s Theorem, $M$ is also a progenerator in $\text{Mod-}R'$.

25B.6. The separability idempotent $e$ is indeed an idempotent, and $(r \otimes 1)e = (1 \otimes r)e$ for all $r \in R$. Conversely, if there exists an idempotent $e \in R^e$ satisfying this condition, then $R$ is separable over $C$, and $e$ is a separability idempotent of $R$.

25B.9. If a module $P$ over a separable $C$-algebra $R$ is projective as a $C$-module, then $P$ is projective as an $R$-module.

25B.10. If $R$ is separable over a field $F$, then $R$ is separable in the classical sense; i.e., $R$ is semisimple and $R \otimes_F \bar{F}$ is semisimple where $\bar{F}$ is the algebraic closure of $F$.

25B.15. If $R$ is separable over its center $C$, then any maximal ideal $B$ of $R$ has the form $AR$, where $A = B \cap C \triangleleft C$, and $R/AR$ is central simple over the field $C/A$.

25B.17. Equivalent conditions for a $C$-algebra $R$ to be Azumaya.

25B.20 (Artin-Procesi). A $C$-algebra $R$ is Azumaya of rank $n^2$ iff $R$ satisfies all polynomial identities of $M_n(\mathbb{Z})$, and no homomorphic image of $R$ satisfies the standard identity $s_{2n-2}$. (Other equivalent PI-conditions are also given.)

25C.8. Any basic f.d. algebra with $J^2 = 0$ is a homomorphic image of the path algebra $\mathcal{P}(R)$.

25C.11 (Gabriel). Suppose $R$ is a f.d. algebra over an algebraically closed field and $J^2 = 0$. Then $R$ has finite representation type iff its quiver (viewed as an undirected graph) is a disjoint union of Dynkin diagrams of types $A_n, D_n, E_6, E_7,$ or $E_8$.

25C.17. Any $F$-subalgebra $R$ of $M_n(F)$ can be put into block upper triangular form (with respect to a suitable change of base of $F^{(n)}$).

E25.6. Every submodule of a projective module over a hereditary ring is projective.

E25.7. A fractional ideal $P$ of an integral domain $C$ is invertible (as a fractional ideal) iff $P$ is projective as a module.

E25.9, E25.10 (Bourbaki). An example of a module that is invertible and thus projective, but not principal.

E25.17. Equivalent conditions for a module over a commutative ring to be invertible.
E25.20 (Schanuel’s Lemma). If $0 \to K_i \to P_i \to M \to 0$ are exact with $P_i$ projective for $i = 1, 2$, then $P_1 \oplus K_2 \cong P_2 \oplus K_1$.


E25.24. $\text{pd}_{R[\lambda]} M \leq \text{pd}_R M + 1$ for any $R[\lambda]$-module $M$.

E25.25. (Eilenberg). For any projective module $P$, the module $P \oplus F$ is free for some free module $F$.

E25.28. (Baer’s criterion). To verify injectivity, it is enough to check Equation (25.5) for $M = R$.

E25.37. $P^* = \text{Hom}_C(P, E)$ is injective, for any flat right $R$-module $P$ and any injective $C$-module $E$.

E25.39. Any module has an injective hull.

E25.45. For any adjoint pair $(F, G)$ of functors, $F$ is right exact and $G$ is left exact.

E25.47. Any homological $\delta$-functor defined by a bifunctor is independent of the choice of component.

E25.53. The homology functor is a universal $\delta$-functor.

E25.55 (Generic flatness). If $S^{-1}M$ is free as an $S^{-1}C$-module, then there is $s \in S$ such that $M[s^{-1}]$ is free as a $C[s^{-1}]$-module.

E25.56. Every finitely presented flat module is projective.

E25.58. Group algebras over a field are quasi-Frobenius.

E25.61. $\text{gl dim } R = \sup \{ n : \text{Ext}^n(M, N) \neq 0 \text{ for all } R\text{-modules } M, N \} = \sup \{ \text{injective dimensions of all } R\text{-modules} \}$.

E25.65. $\text{Ext}^1(M, N)$ can be identified with the equivalence classes of module extensions $0 \to N \to E \to M \to 0$.

E25.69. The corestriction map is compatible with the transfer in the cohomology of $H^2(G, K^\times)$.

E25.71. $H^1(L, M) = \text{Deriv}(L)/\text{InnDeriv}(L)$ for any Lie algebra $L$.

E25A.6. Morita equivalent commutative rings are isomorphic.

E25A.9. Properties of Morita contexts with $\tau, \tau'$ onto.
E25B.6. If $H^2(R,\_)=0$ and $R$ has a nilpotent ideal $N$ such that $R/N$ is separable, then $R$ has a subalgebra $S \cong R/N$ that is a complement to $N$ as a $C$-module. 591

E25B.11. $0 \to M^R \to M \to \text{Deriv}_C(R,M) \to \text{Ext}^1_{R^e}(R,M) \to 0$ is an exact sequence. 592

E25B.12. Equivalent conditions for an algebra to be separable, in terms of derivations. 593

E25B.14 (Braun). A $C$-algebra $R$ is Azumaya, iff there are $a_i,b_i \in R$ such that $\sum a_i b_i = 1$ and $\sum a_i R b_i \subseteq C$. 593

E25B.17. Any Azumaya algebra is a finite direct product of algebras of constant rank when the base ring has no nontrivial idempotents. 593

**Chapter 26.**

26.21 (The Fundamental Theorem of Hopf Modules). Any Hopf module $M$ is isomorphic to $H \otimes M^{co H}$ as Hopf modules (the latter under the “trivial action” $h'(h \otimes a) = (h'h \otimes a)$). 558

26.28 (Nichols-Zoeller [NiZ]). If $K$ is a Hopf subalgebra of a f.d. Hopf algebra $H$, then $H$ is free as a $K$-module, and $\dim K \mid \dim H$. 561

26.30. A f.d. Hopf algebra $H$ is semisimple iff $\varepsilon(1_H^1) \neq 0$. 562

E26.3, E26.5. For any algebra $A$ and coalgebra $C$, $\text{Hom}(C,A)$ becomes an algebra under the convolution product ($\ast$). If $H$ is a Hopf algebra, then its antipode $S$ is the inverse to $1_H$ in $\text{Hom}(H,H)$ under the convolution product. $S(ab) = S(b)S(a)$, $\Delta \circ S = \tau \circ (S \otimes S) \circ \Delta$, and $\varepsilon \circ S = \varepsilon$. 594

E26.16 (Fundamental Theorem of Comodules). Any finite subset of a comodule $M$ (over a coalgebra $C$) is contained in a finite-dimensional submodule of $M$. 595

E26.17 (Fundamental Theorem of Coalgebras). Any finite subset of a coalgebra $C$ is contained in a f.d. subcoalgebra of $C$. 595

E26.28. The following equations hold for $R = \sum a_i \otimes b_i$ in a quasi-triangular Hopf algebra: $R^{-1} = \sum S(a_i) \otimes b_i$; $\sum \varepsilon(a_i)b_i = \sum a_i \varepsilon(b_i) = 1$; $(S \otimes S)(R) = R$. 596

E26.32. For any almost cocommutative Hopf algebra $H$ with antipode $S$, there exists invertible $u \in H$ such that $uS(u)$ is central and $S^2$ is the inner automorphism given by conjugation with respect to $u$. 597
E26.38. The smash product naturally gives rise to a Morita context $(A \# H, A^H, A, A', \tau, \tau')$.

E26.40. The quantum groups of Examples 16A.3 are Hopf algebras.
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This book is a companion volume to *Graduate Algebra: Commutative View* (published as volume 73 in this series). The main and most important feature of the book is that it presents a unified approach to many important topics, such as group theory, ring theory, Lie algebras, and gives conceptual proofs of many basic results of noncommutative algebra. There are also a number of major results in noncommutative algebra that are usually found only in technical works, such as Zelmanov’s proof of the restricted Burnside problem in group theory, word problems in groups, Tits’s alternative in algebraic groups, PI algebras, and many of the roles that Coxeter diagrams play in algebra.

The first half of the book can serve as a one-semester course on noncommutative algebra, whereas the remaining part of the book describes some of the major directions of research in the past 100 years. The main text is extended through several appendices, which permits the inclusion of more advanced material, and numerous exercises. The only prerequisite for using the book is an undergraduate course in algebra; whenever necessary, results are quoted from *Graduate Algebra: Commutative View*.

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