Graduate Algebra: Noncommutative View

Louis Halle Rowen

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To the memory of my beloved mother
Ruth Halle Rowen, April 5, 1918 – January 5, 2007

Contents

Introduction

As indicated in the title, this volume is concerned primarily with noncommutative algebraic structures, having grown from a course introducing complex representations of finite groups via the structure of group algebras and their modules. Our emphasis is on algebras, although we also treat some major classes of finite and infinite groups. Since this volume was conceived as a continuation of Volume 1 (*Graduate Algebra: Commutative View*, Graduate Studies in Mathematics, volume 73), the numeration of chapters starts with Chapter 13, Part IV, and we use the basics of rings and modules developed in Part I of Volume 1 (Chapters 1–3). Nevertheless, Chapters 13–15 and 18 can largely be read independently of Volume 1.

In the last one hundred years there has been a vast literature in noncommutative theory, and our goal here has been to find as much of a common framework as possible. Much of the theory can be cast in terms of representations into matrix algebras, which is our major theme, dominating our treatment of algebras, groups, Lie algebras, and Hopf algebras. A secondary theme is the description of algebraic structures in terms of generators and relations, pursued in the appendices of Chapter 17, and leading to a discussion of free structures, growth, word problems, and Zelmanov's solution of the Restricted Burnside Problem.

One main divergence of noncommutative theory from commutative theory is that left ideals need not be ideals. Thus, the important notion of "principal ideal" from commutative theory becomes cumbersome; whereas the principal left ideal Ra is described concisely, the smallest ideal of a noncommutative ring QR containing an element a includes all elements of the form

$$r_{1,1}ar_{1,2} + \cdots + r_{m,1}ar_{m,2}, \quad \forall r_{i,1}, r_{i,2}, \in R,$$

where m can be arbitrarily large. This forces us to be careful in distinguishing "left" (or "right") properties from two-sided properties, and leads us to rely heavily on modules.

There are many approaches to structure theory. We have tried to keep our proofs as basic as possible, while at the same time attempting to appeal to a wider audience. Thus, projective modules (Chapter 25) are introduced relatively late in this volume.

The exposition is largely self-contained. Part IV requires basic module theory, especially composition series (Chapter 3 of Volume 1). Chapter 16 draws on material about localization and Noetherian rings from Chapters 8 and 9 of Volume 1. Chapter 17, which goes off in a different direction, requires some material (mostly group theory) given in the prerequisites of this volume. Appendix 17B generalizes the theory of Gröbner bases from Appendix 7B of Volume 1. Chapter 18 has applications to field theory (Chapter 4 of Volume 1).

Parts V and VI occasionally refer to results from Chapters 4, 8, and 10 of Volume 1. At times, we utilize quadratic forms (Appendix 0A) and, occasionally, derivations (Appendix 6B). The end of Chapter 24 draws on material on local fields from Chapter 12. Chapters 25 and 26 require basic concepts from category theory, treated in Appendix 1A.

There is considerable overlap between parts of this volume and my earlier book, *Ring Theory* (student edition), but the philosophy and organization is usually quite different. In *Ring Theory* the emphasis is on the general structure theory of rings, via Jacobson's Density Theorem, in order to lay the foundations for applications to various kinds of rings.

The course on which this book is based was more goal-oriented — to develop enough of the theory of rings for basic representation theory, i.e., to prove and utilize the Wedderburn-Artin Theorem and Maschke's Theorem. Accordingly, the emphasis here is on semisimple and Artinian rings, with a short, direct proof. Similarly, the treatment of Noetherian rings here is limited mainly to Goldie's Theorem, which provides most of the non-technical applications needed later on.

Likewise, whereas in *Ring Theory* we approached representation theory of groups and Lie algebras via ring-theoretic properties of group algebras and enveloping algebras, we focus in Part V of this volume on the actual groups and Lie algebras.

Thanks to Dror Pak for pointing me to the proofs of the hook categories, to Luda Markus-Epstein for material on Stallings foldings, to Alexei Belov for gluing components in the Wedderburn decomposition, and to Sue Montgomery for a description of the current state of the classification of finite dimensional Hopf algebras. Steve Shnider, Tal Perri, Shai Sarussi, and Luie Polev provided many helpful comments. Again, as with Volume 1, I would like to express special gratitude to David Saltman, in particular for his valuable suggestions concerning Chapter 24 and Chapter 25, and also to Uzi Vishne. Thanks to Sergei Gelfand for having been patient for another two years. And, of course, many thanks to Miriam Beller for much of the technical preparation of the manuscript.

Needless to say, I am deeply indebted to Rachel Rowen, my helpmate, for her steadfast support all of these years.

List of Symbols

Warning: Some notations have multiple meanings.

Prerequisites

As mentioned in the Introduction, most of Part IV of Volume 2 is self-contained, modulo some basic results on rings and modules. In Chapter 17, we need a few extra general basic results, mostly concerning finitely generated groups, which we list here.

Finitely generated (f.g.) groups.

A fair part of Chapter 17 concerns f.g. groups, introduced briefly in Volume 1, namely on p. 13 and Exercises 0.23–0.27. Often we look for f.g. subgroups of a given f.g. group. The following straightforward facts often come in handy. Recall that a subgroup H has **finite index** G if H has finitely many cosets in G, the number of which is designated as $[G{:}H]$.

Remark 00.1. Any subgroup H of finite index in a f.g. group G is also f.g. (This was stated in Exercise 0.27 of Volume 1, with an extensive hint.) The same proof shows, more precisely, that if G is generated by t elements and $[G{:}H] = m$, then H is generated by tm elements.

LEMMA 00.2. *For any $n \in \mathbb{N}$, any f.g. group G has finitely many subgroups of index n.*

Proof. We elaborate on Exercise 0.25 of Volume 1. For any subgroup H of index n, we have a homomorphism $\psi_H\colon G \to S_n$, given by left multiplication on the cosets of H. But any element a of $\ker \psi_H$ satisfies $aH = H$, implying $\ker \psi_H \subseteq H$, and thus $H = \psi_H^{-1}(\overline{H})$ for some subgroup $\overline{H}$ of S_n.

Working backwards, since G is f.g., there are only finitely many homomorphisms from G to S_n, which has finitely many possible subgroups $\overline{H}$. Since any subgroup H of index n can be recovered in this way, we have only finitely many possibilities for H. □

Proposition 00.3. *If H is a f.g. normal subgroup of G, and K is a subgroup of finite index in H, then K contains a f.g. normal subgroup of G that has finite index in H. (The special case for $H = G$ was given in Exercise 0.25 of Volume 1.)*

Proof. For each $g \in G$, gKg^{-1} is a subgroup of $gHg^{-1} = H$ of the same index as K; by the lemma, there are only finitely many of these, so, by Exercise 0.24 of Volume 1, $\bigcap_{g\in G} gKg^{-1}$ is a normal subgroup of G having finite index in H. □

Groups of fractions.

In the proof of Theorem 17.61 we also need the following easy special case of the construction of Exercise 8.26 of Volume 1:

Definition 00.4. Suppose $(A, +)$ is a torsion-free Abelian group. The group $A_{\mathbb{Q}}$ is defined as follows:

Define an equivalence on $A \times \mathbb{N}^+$ by putting $(a, m) \sim (b, n)$, iff $an = bm$. Writing $\frac{a}{m}$ for the equivalence class $[(a, m)]$, we define $A_{\mathbb{Q}}$ to be the set of equivalence classes, endowed with the operation

$$\frac{a}{m} + \frac{b}{n} = \frac{an + bm}{mn}.$$

Remark 00.5. $A_{\mathbb{Q}}$ is a group, and in fact is a $\mathbb{Q}$-module in the natural way, namely

$$\frac{u}{v}\frac{a}{m} = \frac{ua}{vm}, \qquad a \in A,\ u \in \mathbb{Z},\ m, v \in \mathbb{N}^+.$$

There is a group injection $A \to A_{\mathbb{Q}}$ given by $A \mapsto \frac{a}{1}$. Furthermore, any automorphism σ of A extends naturally to an automorphism of $A_{\mathbb{Q}}$ via the action $\sigma(\frac{a}{m}) = \frac{\sigma(a)}{m}$.

(The verifications are along the lines of those in the proof of Proposition 12.18 of Volume 1. Alternatively, once we have tensor products from Chapter 18, we could view $A_{\mathbb{Q}}$ as $A \otimes_{\mathbb{Z}} \mathbb{Q}$.)

Jordan decomposition.

The Jordan decomposition of Theorem 2.75 of Volume 1 has an easy but useful application in nonzero characteristic:

Proposition 00.6. *Over a field of characteristic $p > 0$, any $n \times n$ matrix T has a power whose radical component is* 0.

Proof. Write the Jordan decomposition $T = T_{\mathbf{s}} + T_{\mathbf{n}}$, where the semisimple component $T_{\mathbf{s}}$ and the nilpotent component $T_{\mathbf{n}}$ commute. Then, as in Corollary 4.69 of Volume 1,

$$T^{p^k} = (T_{\mathbf{s}} + T_{\mathbf{n}})^{p^k} = T_{\mathbf{s}}^{p^k} + T_{\mathbf{n}}^{p^k}$$

for each k, but $T_{\mathbf{n}}^{p^k} = 0$ whenever $p^k > n$, so we conclude for such k that $T^{p^k} = T_{\mathbf{s}}^{p^k}$ is semisimple. □

Galois theory.

We also need a fact from Galois theory, which was missed in Volume 1.

PROPOSITION 00.7. *Suppose F is a finite field extension of $\mathbb{Q}$, and $a \in F$ is integral over $\mathbb{Z}$. If $|\sigma(a)| \leq 1$ for every embedding $\sigma: F \to \mathbb{C}$, then a is a root of unity.*

Proof. The minimal monic polynomial $f_a \in \mathbb{Z}[\lambda]$ of a over $\mathbb{Z}$ has some degree n; its coefficients are sums of products of conjugates of a, and so by hypothesis have absolute value $\leq n$. But there are at most $(2n+1)^n$ possibilities for such a polynomial; moreover, the hypothesis also holds for each power of a, which must thus be a root of one of these polynomials. We conclude that there are only finitely many distinct powers of a, which means a is a root of unity. □

The trace bilinear form.

We need a result about the **trace bilinear form** on the matrix algebra $M_n(F)$ over a field F, given by $\langle x, y\rangle = \operatorname{tr}(xy)$. Clearly this form is symmetric and also nondegenerate, for if $x = (a_{ij})$ with $a_{i_0 j_0} \neq 0$, then $\operatorname{tr}(xe_{j_0 i_0}) = a_{i_0 j_0} \neq 0$. The **discriminant** of a base $\mathcal{B} = \{b_1, \dots, b_{n^2}\}$ of $M_n(F)$ is defined as the determinant of the $n^2 \times n^2$ matrix $(\operatorname{tr}(b_i b_j))$. In view of Remark 4B.5 of Volume 1, the discriminant of any base $\mathcal{B}$ is nonzero (since there exists an orthogonal base with respect to the trace bilinear form).

LEMMA 00.8. *Suppose $\{b_1, \dots, b_n\}$ is a base of $M_n(F)$ over F. Then for any $\alpha_1, \dots, \alpha_n^2 \in F$, the system of n^2 equations $\{\operatorname{tr}(b_i x) = \alpha_i : 1 \leq i \leq n^2\}$ has at most one solution for $x \in M_n(F)$.*

Proof. Write $x = \sum_{j=1}^{n^2} \gamma_j b_j$. Then $\alpha_i = \sum_{j=1}^{n^2} \gamma_j \operatorname{tr}(b_i b_j)$, $1 \leq i \leq n^2$, can be viewed as n^2 equations in the γ_j; since the discriminant $\det(\operatorname{tr}(b_i b_j))$ is nonzero, one can solve these equations using Cramer's rule.

To prove uniqueness, suppose there were two matrices x_1 and x_2 such that $\operatorname{tr}(b_i x_1) = \operatorname{tr}(b_i x_2)$, $1 \leq i \leq n^2$. Then $\operatorname{tr}(b_i(x_1 – x_2)) = 0$ for each i, which implies $x_1 - x_2 = 0$ since the trace form is nondegenerate; thus, $x_1 = x_2$. □

List of Major Results

The prefix E denotes that the referred result is an exercise, such as E0.5. Since the exercises do not necessarily follow a chapter immediately, their page numbers may be out of sequence.

Prerequisites.

Chapter 13.

13.42. $M_n(W) \cong (\mathrm{End}_W W^{(n)})^{\mathrm{op}}$ as rings. 22

13.44. $\mathrm{Hom}\left(\bigoplus_{i\in I} M_i, \bigoplus_{j\in J} N_j\right)_W \cong \bigoplus_{i,j} \mathrm{Hom}(M_i, N_j)_W$ as additive groups, for any right W-modules M_i, N_j. 23

13.47. $\mathrm{End}_R(S_1^{(n_1)} \oplus \cdots \oplus S_t^{(n_t)}) \cong \prod_{i=1}^t M_{n_i}(D_i)$, for simple pairwise nonisomorphic simple R-modules S_i, where $D_i = \mathrm{End}\, S_i$. 25

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18.12. Suppose M is a free right R-module with base $B = \{b_i : i \in I\}$, and N is an R-module. Then every element of $M \otimes N$ can be written uniquely in the form $\sum_{i\in I} b_i\otimes v_i$ for v_i in N. 143

18.13. $C^{(m)}\otimes_C C^{(n)} \cong C^{(mn)}$. 144

18.15. $M_1 \otimes_{R_2} (M_2\otimes_{R_3} M_3) \cong (M_1\otimes_{R_2} M_2) \otimes_{R_3} M_3$. 144

18.16. $\tau\colon A \otimes_C B \cong B \otimes_C A$. 145

18.21. If A and B are C-algebras, then $A \otimes_C B$ is also a C-algebra with multiplication $(a\otimes b)(a'\otimes b') = aa' \otimes bb'$ and $c(a\otimes b) = ca \otimes b$. 147

18.25. The following algebra isomorphisms hold for any C-algebras: $A\otimes_C C \cong C\otimes_C A \cong A$; $A_1\otimes A_2 \cong A_2\otimes A_1$; $A_1\otimes(A_2\otimes A_3) \cong (A_1\otimes A_2)\otimes A_3$. 149

18.29′. A finite field extension $K \supseteq F$ is separable iff the ring $K\otimes_F K$ is semisimple. 151

18.31. Any splitting field K of an F-algebra R contains some subfield K_0 f.g. over F such that K_0 also splits R. 152

18.33. If R is simple with center a field F, and W is an F-algebra, then any nonzero ideal I of the tensor product $R \otimes_F W$ contains $1 \otimes w$ for some $w \in W$. In particular, if W is simple, then $R \otimes_F W$ is also simple. 152

18.36. $M_m(C) \otimes M_n(C) \cong M_{mn}(C)$. 153

18.41. The tensor product of two integral domains over an algebraically closed field F is an integral domain. 157

18.42. If X and Y are affine varieties over an algebraically closed field F, then $X \times Y$ is an affine variety, with $F[X] \otimes F[Y] \cong F[X \times Y]$. 157

18.44. $\Phi : \mathrm{Hom}_R(A \otimes_S B, C) \cong \mathrm{Hom}_S(B, \mathrm{Hom}_R(A, C))$. 158

E18.2. $(\bigoplus_{i\in I} M_i) \otimes N \cong \bigoplus_{i\in I}(M_i \otimes N)$. 189

E18.7. If $K \to N \to P \to 0$ is an exact sequence of right R-modules, then $K \otimes M \to N \otimes M \to P \otimes M \to 0$ is also exact. 190

E18.12. $C(V, Q)$ has an involution (*) satisfying $v^* = v$, $\forall v \in V$. 191

E18.16. For any separable field extension K of F, $K \otimes_F K$ has a simple idempotent e with $(a \otimes b)\, e = (b \otimes a)\, e$ for all $a, b \in K$. 191

E18.18 (Wedderburn's Principal Theorem). Any finite-dimensional algebra R over a perfect field F has a Wedderburn decomposition $R = S \oplus J$ for a suitable semisimple subalgebra $S \cong R/J$ of R. 191

E18.19. The tensor product of two reduced algebras over an algebraically closed field is reduced. 191

E18.23 (Amitsur). If R is an algebra without nonzero nil ideals over a field F, then $\mathrm{Jac}(R \otimes_F F(\lambda)) = 0$. 192

E18.24. $K \otimes_F \mathrm{Jac}(R) \subseteq \mathrm{Jac}(K \otimes_F R)$ whenever $K \supseteq F$ are fields and R is an algebra over F, equality holding if K/F is separable. 192

Chapter 19.

19.18. For any vector space V over a field F, there is a 1:1 correspondence among: group representations $\rho: G \to \mathrm{GL}(V)$, algebra representations $F[G] \to \mathrm{End}_F V$, G-space structures on V, and $F[G]$-module structures on V. 206

19.22. A group representation ρ of degree n is reducible iff there is a representation τ equivalent to ρ for which each matrix $\tau(g)$, $g \in G$, has the form (19.4) (for suitable $1 \leq m < n$). 208

19.26 (Maschke's Theorem). $F[G]$ is a semisimple ring, for any finite group G whose order is not divisible by char(F). 210

19.33. Any finite group G has a splitting field that is finite over $\mathbb{Q}$. 212

19.36. For any splitting field F of the group G, a representation ρ of degree n is irreducible iff $\{\rho(g) : g \in G\}$ spans $M_n(F)$. 213

19.38. The following are equivalent, for F a splitting field of a finite group G: (i) G is Abelian; (ii) The group algebra $F[G]$ is commutative; (iii) $F[G] \cong F \times F \times \cdots \times F$; (iv) Every irreducible representation of G has degree 1. 213

19.42. Cent$(C[G])$ is free as a C-module. 215

19.43. The following numbers are equal, for F a splitting field of a finite group G: (i) the number of conjugacy classes of G; (ii) the number of inequivalent irreducible representations of G; (iii) the number of simple components of $F[G]$; (iv) $\dim_F \operatorname{Cent}(F[G])$. 216

19.48. Any complex irreducible representation of G of degree n_i either is extended from a real irreducible representation or corresponds to a real irreducible representation of degree $2n_i$. 218

19.61. If char$(F) = 0$ or char$(F) > n$, then $I_\lambda = \bigoplus_{T_\lambda \text{ standard}} F[S_n]e_{T_\lambda}$. 223

19.64 (Frame, Robinson, and Thrall). $f^\lambda = \frac{n!}{\prod h_{i,j}}$. 226

19A.4. If $\{(a,b) : a \in A,\, b \in B\}$ is finite for $A, B \lhd G$, then the group (A, B) is finite. 229

19A.9 (Burnside, Schur). In characteristic 0, every linear group of finite exponent is finite, and any f.g. periodic linear group is finite. 231

19A.12. Every open subgroup of a quasicompact group is closed of finite index. 234

19A.16. Every continuous f.d. representation of a compact (Hausdorff) group is a finite direct sum of continuous irreducible representations. 235

19A.19. Any Lie homomorphism $\phi\colon G \to H$ of Lie groups (G connected) is uniquely determined by its tangent map $d_e\phi$. 236

19B.4. In any algebraic group G, each open subgroup of G is closed of finite index, each closed subgroup H of G of finite index is open, and G_e is clopen of finite index in G. 239

19B.11. If $H \leq G$, then $\overline{H} \leq G$; furthermore, if H contains a nonempty open subset U of $\overline{H}$, then H is closed. 240

19B.19. Every affine algebraic group is linear. 243

19B.21 (The Tits alternative). Every f.g. linear group either is virtually solvable or contains a free subgroup. 244

19B.24 (Breuillard-Gelanter). Any f.g. linear group contains either a free subgroup that is Zariski dense (in the relative topology), or a Zariski open solvable subgroup. 248

E19.6 (Schur's Lemma, representation-theoretic formulation). For F a splitting field for G, $\mathrm{End}_{F[G]}(L_i) \cong F$ and $\mathrm{Hom}_{F[G]}(L_i, L_j) = 0$ for all $i \neq j$, where L_i denotes the module corresponding to ρ_i. 355

E19.13. A representation ρ of finite degree ρ is completely reducible whenever its G-space has a G-invariant Hermitian form. 356

E19.31. $C[G]$ is semiprime, for any group G and any integral domain C of characteristic 0. 358

E19.34 (Herstein; Amitsur). $\mathrm{Jac}(F[G]) = 0$ for any uncountable field F of characteristic 0. 358

E19.42 (Schur's Double Centralizer Theorem.) Suppose V is any f.d. vector space over a field of characteristic 0. The diagonal action of $\mathrm{GL}(V)$ and the permutation action of S_n on $V^{\otimes n} = V \otimes \cdots \otimes V$ centralize each other, and provide algebra homomorphisms $\hat{\rho}\colon F[\mathrm{GL}(V)] \to \mathrm{End}_F V^{\otimes n}$ and $\hat{\tau}\colon F[S_n] \to \mathrm{End}_F V^{\otimes n}$. Their respective images are the centralizers of each other in $\mathrm{End}_F V^{\otimes n}$. 359

E19A.6 (Burnside). Any f.g. periodic linear group is finite. 361

E19A.8 (Schur). Each periodic subgroup of $\mathrm{GL}(n, \mathbb{C})$ consists of unitary matrices with respect to some positive definite Hermitian form. 361

E19A.11 (Jordan). Any unitary subgroup $G \subseteq \mathrm{GL}(n, \mathbb{C})$ has a normal Abelian subgroup of index bounded by $(\sqrt{8n} + 1)^{2n^2} - (\sqrt{8n} - 1)^{2n^2}$. 362

E19A.16. For any continuous complex representation of degree n of a compact topological group G, the vector space $\mathbb{C}^{(n)}$ has a positive definite G-invariant Hermitian form. 362

E19A.18. Any continuous f.d. representation of G having a positive definite G-invariant Hermitian form is completely reducible 363

E19A.30 (Artin's combing procedure). The kernel of the map $P_n \to P_{n-1}$ obtained by cutting the n-th strand is the free group of rank $n-1$. 364

E19A.34. The braid group B_n satisfies $B_n' = (B_n', B_n)$. 364

E19B.7. The Tits alternative also over fields of any characteristic. 366

E19B.14. The commutator group of two closed subgroups of an algebraic group G is closed. In particular, all the derived subgroups of G are closed, and all subgroups in its upper central series are closed. 367

E19B.16. For F algebraically closed, any connected solvable algebraic subgroup G of $\mathrm{GL}(n,F)$ is conjugate to a subgroup of $\mathrm{T}(n,F)$. 367

Chapter 20.

20.5. The characters $\chi_1,\dots,\chi_t$ comprise an orthonormal base of $\mathcal{R}$ with respect to the Schur inner product. 251

20.10. $\sum_{g\in G}\chi_i(ga)\overline{\chi_j(g)} = \frac{\delta_{ij}|G|\chi_i(a)}{n_i}$ for each $a \in G$. 252

20.14 (Schur I). $\delta_{ik}|G| = \sum_{j=1}^t m_j\chi_{ij}\overline{\chi_{kj}}$. 256

20.15 (Schur II). $\sum_{i=1}^t \chi_{ij}\overline{\chi_{ik}} = \delta_{jk}\,\frac{|G|}{m_k}$. 256

20.18 (Frobenius). n_i divides $|G|$ for each i. 258

20.20. If $\gcd(n_i,m_j)=1$, then either $\chi_{ij}=0$ or $|\chi_{ij}| \ = \ n_i$. 259

20.22. In a finite nonabelian simple group, the size of a conjugacy class cannot be a power (other than 1) of a prime number. 259

20.24 (Burnside). Every group of order p^uq^v (p,q prime) is solvable.260

20.32. The character table of $G\times H$ is the tensor product of the character tables of G and of H. 262

20.42 (Frobenius Reciprocity Theorem). For $F\subseteq\mathbb{C}$ a splitting field of a finite group G, if σ is an irreducible representation of a subgroup H and ρ is an irreducible representation of G, then the multiplicity of ρ in σ^G is the same as the multiplicity of σ in ρ_H. 267

20.43. For $H<K<G$ and a representation ρ of H, the representations $(\rho^K)^G$ and ρ^G are equivalent, $(\rho_1\oplus\rho_2)^G$ and $\rho_1^G\oplus\rho_2^G$, are equivalent, and $\rho^G\otimes\sigma$ and $(\rho\otimes\sigma_H)^G$ are equivalent for any representation σ of G. 268

20.44 (Artin). Every complex character of a group is a linear combination (over $\mathbb{Q}$) of complex characters induced from cyclic subgroups. 269

E20.20. n_i divides $[G:Z_i]$ for each i. 370

E20.22. The degree of each irreducible character of G divides $[G:A]$ for any Abelian normal subgroup A. 370

E20.27. For any representation ρ of finite degree of a subgroup $H \subseteq G$, the contragredient $(\rho^G)^*$ of the induced representation is equivalent to the induced representation $(\rho^*)^G$. 371

Chapter 21.

21.21. For F algebraically closed, if L is a Lie subalgebra of $\subseteq gl(n,F)$ and $a = \mathbf{s} + \mathbf{n}$ is the Jordan decomposition of $a \in L$, then $\mathrm{ad}_a = \mathrm{ad}_{\mathbf{s}} + \mathrm{ad}_{\mathbf{n}}$ is the Jordan decomposition of ad_a. 281

21.27. If L is a Lie subalgebra of R^- and ad_a is nilpotent for every $a \in L$, then $\mathrm{ad}\, L$ is nilpotent under the multiplication of R, and L is a nilpotent Lie algebra. 283

21.29 (Engel). Any Lie algebra $L \subseteq gl(n,F)$ of nilpotent transformations becomes a Lie subalgebra of the algebra of strictly upper triangular matrices under a suitable choice of base. 284

21.32 (Lie). If a Lie subalgebra L of $gl(n,F)$ acts solvably on $F^{(n)}$, with F an algebraically closed field, then L acts in simultaneous upper triangular form with respect to a suitable base of $F^{(n)}$. 285

21.38. If $L \subseteq gl(n,F)$ in characteristic 0 such that $\mathrm{tr}(aL') = 0$ for all $a \in L$, then L' is a nilpotent Lie algebra. 287

21.41 (Cartan's first criterion). A f.d. Lie algebra L of characteristic 0 is solvable iff its Killing form vanishes identically on L'. 288

21.47 (Cartan's second criterion). A f.d. Lie algebra L of characteristic 0 is semisimple iff its Killing form is nondegenerate. 289

21.51. Any f.d. semisimple Lie algebra L of characteristic 0 is a direct sum $\bigoplus S_i$ of simple nonabelian Lie subalgebras S_i, with each $S_i \triangleleft L$, and any Lie ideal of L is a direct sum of some of the S_i. 290

21.53. The trace bilinear form of any representation ρ of a f.d. semisimple Lie algebra is nondegenerate. 290

21.54 (Zassenhaus). Every derivation of a f.d. semisimple Lie algebra L of characteristic 0 is inner. 291

21.57. The Casimir element satisfies $\mathrm{tr}(c_\rho) = n$ and $[\rho(L), c_\rho] = 0$. 292

21.58 (Weyl). Any f.d. representation of a f.d. semisimple Lie algebra L (of characteristic 0) is completely reducible. 292

21.61. For any given nilpotent Lie subalgebra N of a f.d. Lie algebra L, there exists a unique root space decomposition $L = \bigoplus_{\mathbf{a}} L_{\mathbf{a}}$. 295

21.64. $L_{\mathbf{b}} \perp L_{\mathbf{a}}$ for any roots $\mathbf{a} \neq -\mathbf{b}$. 296

21.71, 21.72. Any f.d. semisimple Lie algebra over an algebraically closed field of characteristic 0 has a Cartan subalgebra $\mathfrak{h}$, which is its own nullspace under the corresponding root space decomposition. The restriction of the Killing form to $\mathfrak{h}$ is nondegenerate. $\mathfrak{h}$ is Abelian, and ad_h is semisimple for all $h \in \mathfrak{h}$. 298

21.79. For any root $\mathbf{a}$, $\dim L_{\mathbf{a}} = \dim L_{-\mathbf{a}} = 1$, and $k\mathbf{a}$ is not a root whenever $1 < |k| \in \mathbb{N}$. 301

21.80. $\langle h_1, h_2 \rangle = \sum_{\mathbf{a} \neq 0} \mathbf{a}(h_1)\mathbf{a}(h_2), \quad \forall h_1, h_2 \in \mathfrak{h}$. 301

21.84. Any simple $\hat{L}_{\mathbf{a}}$-module V has an eigenspace decomposition $V = V_m \oplus V_{m-2} \oplus \cdots \oplus V_{-(m-2)} \oplus V_{-m}$, where each component $V_{m-2j} = Fv_j$ is a one-dimensional eigenspace of $h_{\mathbf{a}}$ with eigenvalue $m - 2j$. In particular, V is determined up to isomorphism by its dimension $m + 1$. 303

21.88. $[L_{\mathbf{a}} L_{\mathbf{b}}] = L_{\mathbf{b}+\mathbf{a}}$ whenever $\mathbf{a}, \mathbf{b}$, and $\mathbf{b} + \mathbf{a}$ are roots. 305

21.91. $\langle \mathbf{a}, \mathbf{a} \rangle > 0$ and $\langle \mathbf{a}, \mathbf{b} \rangle \in \mathbb{Q}$ for all nonzero roots $\mathbf{a}, \mathbf{b}$. The bilinear form given by Equation (21.18) restricts to a positive form on ${\mathfrak{h}^*}_0$, the $\mathbb{Q}$-subspace of $\mathfrak{h}^*$ spanned by the roots, and $\mathfrak{h}^* = {\mathfrak{h}^*}_0 \otimes_{\mathbb{Q}} F$. 306

21.96. $\langle \mathbf{a}, \mathbf{b} \rangle \leq 0$ for all $\mathbf{a} \neq \mathbf{b} \in P$. 308

21.97. The set of simple roots is a base of the vector space V and is uniquely determined by the given order on V. 308

21.102. The Cartan numbers m_{ij} satisfy $m_{ij} m_{ji} < 4$. 310

21.103. The Cartan numbers are integers. 311

21.108. Suppose $S = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is a simple root system for the semisimple Lie algebra L. Take $e_i \in L_{\mathbf{a}_i}$, $e_i' \in L_{-\mathbf{a}_i}$, and $h_i = [e_i f_i]$. Writing any positive root $\mathbf{a} = \mathbf{a}_{i_1} + \cdots + \mathbf{a}_{i_\ell}$, let $x_{\mathbf{a}} = [e_{i_1} e_{i_2} \cdots e_{i_\ell}]$ and $y_{\mathbf{a}} = [f_{i_1} f_{i_2} \cdots f_{i_\ell}]$. Then $\{h_1, \ldots, h_n\}$ together with the $x_{\mathbf{a}}$ and $y_{\mathbf{a}}$ comprise a base of L. 313

21.110. The Lie multiplication table of L (with respect to the base in Theorem 21.108) has rational coefficients. 314

21.111. The split f.d. semisimple Lie algebra L is simple iff its simple root system is indecomposable. 315

21.115, 21.116. Any indecomposable generalized Cartan matrix A is of finite, affine, or indefinite type. The symmetric bilinear defined by A is positive definite iff A has finite type, and is positive semidefinite (of corank 1) iff A has affine type. 317

21B.18, 21B.19. Suppose the composition algebra $(\mathcal{A}, *)$ is the ν-double of $(A, *)$. If A is associative, then $\mathcal{A}$ is alternative. If $\mathcal{A}$ is associative, then A must be commutative. 326

21B.22. (Herstein). If R is a simple associative algebra, with $\frac{1}{2} \in R$, then R^+ is simple as a Jordan algebra. 328

21C.5. (PBW Theorem). The map $\nu_L \colon L \to U(L)^-$ is 1:1. 333

E21.10. In characteristic $\neq 2$, the classical Lie algebra B_n is simple for each $n \geq 1$, and C_n and D_n are simple Lie algebras for all $n > 2$. 372

E21.27 (Herstein). For any associative simple ring R of characteristic $\neq 2$, the only proper Lie ideals of R' are central. 373

E21.28 (Herstein). If T is an additive subgroup of a simple ring R of characteristic $\neq 2$ such that $[T, R'] \subseteq T$, then either $T \supseteq R'$ or $T \subseteq Z$. 374

E21.41. The radical of a Lie algebra is contained in the radical of the trace bilinear form with respect to any representation. 375

E21.42. The Casimir element of an irreducible Lie representation is always invertible. 375

E21.44 (Whitehead's First Lemma). For any f.d. Lie module V and linear map $f \colon L \to V$ satisfying $f([ab]) = af(b) - bf(a)$, $\forall a, b \in L$, there is $v \in V$ such that $f(a) = av$, $\forall a \in L$. 376

E21.47 (Whitehead's Second Lemma). For any f.d. semisimple Lie algebra L of characteristic 0 and f.d. Lie module V with $f \colon L \times L \to V$ satisfying $f(a, a) = 0$ and $\sum_{i=1}^{3} f(a_i, [a_{i+1}, a_{i+2}]) + a_i f(a_{i+1}, a_{i+2}) = 0$, subscripts modulo 3, there is a map $g \colon L \to V$ with $f(a_1, a_2) = a_1 g(a_2) - a_2 g(a_1) - g([a_1 a_2])$. 376

E21.48 (Levi's Theorem). Any f.d. Lie algebra L of characteristic 0 can be decomposed as vector spaces $L = S \oplus I$, where $I = \operatorname{rad} L$ and $S \cong L/I$ is a semisimple Lie subalgebra. 377

E21B.9 (Moufang). Every alternative algebra satisfies the three identities $a\big(b(ac)\big) = (aba)c$, $c\big(a(ba)\big) = c(aba)$, and $(ab)(ca) = a(bc)a$. 386

E21B.11 (E. Artin). Any alternative algebra generated by two elements is associative. 386

E21B.19. Any composition F-algebra must be either F itself, the direct product of two copies of F (with the exchange involution), a quadratic field extension of F, a generalized quaternion algebra, or a generalized octonion algebra. 387

E21B.20 (Hurwitz). If $\mathbb{C}$ satisfies an identity $\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} z_i^2$, where z_i are forms of degree 2 in the x_i and y_j, then $n = 1, 2, 4$, or 8. 387

E21B.22 (Zorn). Every f.d. simple nonassociative, alternative algebra is a generalized octonion algebra. 388

E21B.26. The Peirce decomposition of an alternative algebra in terms of pairwise orthogonal idempotents. 388

E21B.29. Any simple alternative algebra A containing three pairwise orthogonal idempotents e_1, e_2, and e_3 is associative. 388

E21B.37 (Glennie). Any special Jordan algebra satisfies the Glennie identity. 389

E21B.39. (Herstein). $\mathcal{S}(R, *)$ is Jordan simple for any simple associative algebra with involution of characteristic $\neq 2$. 390

E21C.4. $U(L)$ is an Ore domain, for any Lie algebra L of subexponential growth. 391

E21C.13 (Ado). Any f.d. Lie algebra of characteristic 0 is linear. 392

E21C.17. $U_q(sl(2, F))$ is a skew polynomial ring. 393

E21C.21. $U_q(L)$ is a Noetherian domain, for any f.d. semisimple Lie algebra L of characteristic 0. 394

Chapter 22.

22.11. Any connected Dynkin diagram is either A_n, $B_n = C_n$, D_n, E_6, E_7, E_8, F_4, or G_2 of Example 22.2. 342

22.13. If any single vertex of the extended Dynkin diagram of a simple affine Lie algebra is erased, the remaining subdiagram is a disjoint union of Dynkin diagrams (of finite type). 345

22.22. For $i \neq j$, the Coxeter bilinear form restricts to a positive semi-definite form on the two-dimensional subspace $Fe_i + Fe_j$, which is positive definite iff $\circ(\sigma_i\sigma_j) < \infty$. 348

22.25. The only abstract Coxeter graphs whose quadratic forms are positive definite are A_n, D_n, E_6, E_7, and E_8. 350

22.28. (Bernstein, Gel′fand, and Ponomarev). If an abstract Coxeter graph $(\Gamma; \nu)$ has only finitely many nonisomorphic indecomposable representations, then its quadratic form is positive definite. 352

E22.1–E22.4. Construction of the classical Lie algebras from their Dynkin diagrams. 394

E22.10. For any generalized Cartan matrix A of affine type, any proper subdiagram of its Dynkin diagram is the disjoint union of Dynkin diagrams of simple f.d. Lie algebras. 396

E22.19. Any finite reflection group is Coxeter. 397

E22.20. Any two positive systems Φ_1 and Φ_2 are conjugate under some element of the Weyl group. 398

E22.23. Each $m_{i,j} \in \{2, 3, 4, 6\}$ for any crystallographic group. 398

E22.26. The bilinear form of any finite Coxeter group $\mathcal{W}$ is positive definite. 398

E22.36. Every finite Coxeter group is a reflection group. 400

Chapter 23.

23.11. Any t-alternating polynomial f is an identity for every algebra spanned by fewer than t elements over its center. 410

23.26 (Razmyslov). There is a 1:1 correspondence between multilinear central polynomials of $M_n(F)$ and multilinear 1-weak identities that are not identities. 415

23.31 (Kaplansky). Any primitive ring R satisfying a PI of degree d is simple of dimension n^2 over its center, for some $n \leq \left[\frac{d}{2}\right]$. 418

23.32. A semiprime PI-ring R has no nonzero left or right nil ideals.418

23.33. Any semiprime PI-ring R has some PI-class n, and every ideal A intersects the center nontrivially. 419

23.34 (Posner et al). The ring of central fractions of a prime PI-ring R is simple and f.d. over the field of fractions of $\mathrm{Cent}(R)$. 419

23.35. Extension of Theorem 15.23 to PI-rings. 419

23.39. Suppose R has PI-class n and center C, and $1 \in h_n(R)$. Then R is a free C-module of rank n; also, there is a natural 1:1 correspondence between {ideals of R} and {ideals of C}. 421

23.48. If R is an algebra over an infinite field F, and H is any commutative F-algebra, then R is PI-equivalent to $R \otimes_F H$. 425

23.51. The algebra of generic matrices is the relatively free PI-algebra with respect to $\mathcal{I} = \mathcal{M}_{n,C}$. 426

23.57. Suppose R satisfies a PI of degree d, and $\frac{1}{u} + \frac{1}{v} \leq \frac{2}{e(d-1)^4}$, where $e = 2.71828\cdots$. Then any multilinear polynomial of a Young tableau whose shape contains a $u \times v$ rectangle is an identity of R. 429

23A.3 (Shirshov's Dichotomy Lemma). For any ℓ, d, k, there is $\beta \in \mathbb{N}$ such that any word w of length $\geq \beta$ in ℓ letters is either d-decomposable or contains a repeating subword of the form u^k with $1 \leq |u| \leq d$. 431

23A.5. Any hyperword h is either d-decomposable or has the form vu^∞ for some initial subword v and some subword u with $|u| < d$. 431

23A.6 (Shirshov's First Theorem). If $R = C\{r_1, \dots, r_\ell\}$ satisfies a PI, and each word in the r_i of length $\leq d$ is integral over C, then R is f.g. as a C-module. 432

23A.7. If R is affine without 1 and satisfies a PI of degree d, and if each word in the generators of length $\leq d$ is nilpotent, then R is nilpotent. 432

23A.10. Any prime PI-algebra and its characteristic closure have a common nonzero ideal. 433

23A.11 (Kemer). Any affine PI-algebra over a field F of characteristic 0 is PI-equivalent to a finite-dimensional algebra. 434

23A.19. For any PI algebra R, the following assertions are equivalent for any multilinear polynomial f of degree n: $f \in \mathrm{id}(R)$; $f_I{}^* \in \mathrm{id}_2(R \otimes G)$ for some subset $I \subseteq \{1, \dots, n\}$; $f_I{}^* \in \mathrm{id}_2(R \otimes G)$ for every subset of $\{1, \dots, n\}$. 436

23A.22 (Kemer). Let R be a PI-superalgebra, and $f = f(x_1, \dots, x_n) = \sum_{\pi \in S_n} \alpha_\pi x_{\pi 1} \cdots x_{\pi n}$. Then $f \in \mathrm{id}(\mathcal{G}(R))$ iff $f_I{}^* \in \mathrm{id}_2(R)$ for every subset $I \subseteq \{1, \dots, n\}$. 437

23A.23 (Kemer). There is a 1:1 correspondence from {varieties of superalgebras} to {varieties of algebras} given by $R \mapsto \mathcal{G}(R)$. 437

23B.5 (Kostrikin-Zelmanov). Over a field of characteristic p, any f.g. Lie algebra satisfying the Engel identity e_{p-1} is Lie nilpotent. 442

23B.6 (Zelmanov). If a f.g. restricted Lie algebra L over a field of characteristic p satisfies the Engel identity e_n and all of its partial linearizations, then L is Lie nilpotent. 442

23B.13. The Lie algebra L of a nilpotent group G is indeed a Lie algebra and is $\mathbb{N}$-graded in the sense that $[L_iL_j] \subseteq L_{i+j}$. L is Lie nilpotent of the same index t as the nilpotence class of the group G. 444

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E23.4. Any algebra that is f.g. as a module over a commutative affine subalgebra is representable. 563

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E23.16. Every identity of an algebra over a field of characteristic 0 is a consequence of its multilinearizations. 565

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E23.26 (Bell). Every prime affine PI-algebra has a rational Hilbert series. 566

E23.30 (Amitsur). Any PI-algebra R satisfies an identity s_d^k. 566

E23.32. If algebras R_1 and R_2 are PI-equivalent, then so are $M_n(R_1)$ and $M_n(R_2)$. 567

E23.36 (Regev). In characteristic 0, the T-ideal $\mathrm{id}(G)$ is generated by the Grassmann identity. 567

E23.40 (Regev). $M_n(G(p))$ satisfies the identity $s_{2n}^{n^2p+1}$. 568

E23.42 (Kemer). In any F-algebra, a suitable finite product of T-prime T-ideals is 0. Any T-ideal has only finitely many T-prime T-ideals minimal over it. 568

E23B.1. Any simple alternative, nonassociative algebra satisfies the central polynomial $[x, y]^2$. 569

E23B.4. Any Lie algebra of characteristic 3 satisfying the Engel ad-identity $e_2 = X^2$ is Lie nilpotent of class ≤ 3. 570

E23B.6. For any nilpotent p-group G of exponent $n = p^k$, the Lie algebra $L_{\hat{\gamma}}(G)$ satisfies the multilinearized n-Engel identity $\tilde{e}_n$ and some weak Engel condition $e_{S,2n}$. 570

E23B.14 (Key step in proving Theorem 23B.16). An enveloping algebra R of a Lie algebra L is nilpotent whenever R is generated by a finite set of 1-thick sandwiches. 571

E23B.17 (Zelmanov). If a f.g. restricted Lie algebra L satisfies various Engel-type conditions, then its associative enveloping algebra R (without 1) is nilpotent. 572

Chapter 24.

24.10. If R_1 and R_2 are csa's, then $R_1 \otimes_F R_2$ is also a csa. 452

24.14. If R is a csa, then $\Phi: R \otimes_F R^{\mathrm{op}} \to \mathrm{End}_F\, R$ is an isomorphism. 453

24.15. The Brauer group $\mathrm{Br}(F)$ is a group, where $[R]^{-1} = [R]^{\mathrm{op}}$. 453

24.23, 24.24. $\mathrm{End}_K\, R \cong C_R(K) \otimes_F R^{\mathrm{op}}$ as K-algebras, for any F-subfield K of R. $C_R(K)$ is a K-csa and $[C_R(K) : F] = [R : K]$. 455, 456

24.25. $R \otimes_F K \sim C_R(K)$ in $\mathrm{Br}(K)$. 456

24.32 (Double Centralizer Theorem). $C_R(K) \cong A \otimes_K C_R(A)$ and $[A : F][C_R(A) : F] = n^2$, for any simple F-subalgebra A of a csa R, where $K = \mathrm{Cent}(A)$, 458

24.34 (Index Reduction Theorem). The index reduction factor divides the g.c.d. of $\mathrm{ind}(R)$ and $m = [L : F]$. 459

24.40 (Skolem-Noether Theorem). Suppose A_1 and A_2 are isomorphic simple subalgebras of a csa R. Any F-algebra isomorphism $\varphi: A_1 \to A_2$ is given by conjugation by some $u \in R^{\times}$. 460

24.42 (Wedderburn). Every finite division ring is a field. 461

24.44. A csa R of degree n over an infinite field F is split iff R contains an element of degree n whose minimal polynomial has a linear factor. 462

24.48′. $(K, \sigma, \beta_1) \otimes (K, \sigma, \beta_2) \sim (K, \sigma, \beta_1\beta_2)$. 464

24.50. Any F-csa R is PI-equivalent to $M_n(F)$ for $n = \deg R$. 465

24.51 (Koethe-Noether-Jacobson). Any separable subfield L of a cda D is contained in a separable maximal subfield of D. 465

24.52. Every csa is similar to a crossed product. 466

24.54. $\mathrm{UD}(n, F)$ is a division algebra of degree n (over its center) for every n and every field F of characteristic prime to n. 467

24.57. If D is a cda of degree $p^u q$ with p prime, $p \nmid q$, then there is a field extension L of F with $p \nmid [L : F]$, as well as a splitting field $L_u \supseteq L$ of D together with a sequence of subfields $L_0 = L \subset L_1 \subset L_2 \subset \cdots \subset L_u$ for which $\mathrm{ind}(D \otimes_F L_i) = p^{u-i}$ for each $0 \le i \le u$, and each L_i/L_{i-1} is cyclic Galois of dimension p. 468

24.62. $\exp(R)$ divides $\mathrm{ind}(R)$. If a prime number p divides $\mathrm{ind}(R)$, then p divides $\exp(R)$. 469

24.66. Any cda D is isomorphic to the tensor product of cda's of prime power index. 470

24.68 (Wedderburn). Suppose D is a cda. If $a \in D$ is a root of a monic irreducible polynomial $f \in F[\lambda]$ of degree n, then $f = (\lambda - a_n) \cdots (\lambda - a_1)$ in $D[\lambda]$, where each a_i is a conjugate of a. 472

24.73, 24.74. For any Galois extension E of F, $\mathrm{cor}_{E/F}$ induces a homomorphism of Brauer groups, and $\mathrm{cor}_{E/F} \mathrm{res}_{E/F} R \cong R^{\otimes [E:F]}$. 475

24.82 (Cohn-Wadsworth). A cda D has a valuation extending a given valuation v on F, iff v extends uniquely to a valuation of each maximal subfield of D. 480

24.85 (Hasse). Any cda D of degree n over a local field is a cyclic algebra, having a maximal subfield K isomorphic to the unramified extension of F of dimension n. 481

E24.1 (Frobenius). The only $\mathbb{R}$-cda other than R is $\mathbb{H}$. 572

E24.8 (Wedderburn's criterion). A cyclic algebra (K, σ, β) of degree n has exponent n, if β^j is not a norm from K for all $1 \le j < n$. 573

E24.25. $(K, G, (c_{\sigma,\tau})) \otimes (K, G, (d_{\sigma,\tau})) \sim (K, G, (c_{\sigma,\tau} d_{\sigma,\tau}))$. 575

E24.31. Any p-algebra is split by a purely inseparable, finite-dimensional field extension. 575

25.38 (The Snake Lemma). Any commutative diagram

$$\begin{array}{ccccccc} & & A_1'' & \xrightarrow{f_1} & A_1 & \xrightarrow{g_1} & A_1' & \longrightarrow & 0 \\ & & {\scriptstyle d''}\downarrow & & {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d'} & & \\ 0 & \longrightarrow & A_2'' & \xrightarrow{f_2} & A_2 & \xrightarrow{g_2} & A_2' & & \end{array}$$

gives rise to an exact sequence $\ker d'' \to \ker d \to \ker d' \to \operatorname{coker} d'' \to \operatorname{coker} d \to \operatorname{coker} d'$. 506

25.44. For any exact sequence $0 \to M' \to M \to M'' \to 0$ of modules and respective projective resolutions $(\mathbb{P}', d')$ and $(\mathbb{P}'', d'')$ of M' and M'', there exists a projective resolution $(\mathbb{P}, d)$ of M, such that $P_n = P_n' \oplus P_n''$ for each n, and the three projective resolutions form a commutative diagram. 509

25.45. Any short exact sequence $0 \to (\mathbf{A}'', d'') \xrightarrow{f} (\mathbf{A}, d) \xrightarrow{g} (\mathbf{A}', d') \to 0$ of complexes gives rise to a long exact sequence of the homology groups $\cdots \to H_{n+1}(\mathbf{A}'') \xrightarrow{f_*} H_{n+1}(\mathbf{A}) \xrightarrow{g_*} H_{n+1}(\mathbf{A}') \xrightarrow{\partial_*} H_n(\mathbf{A}'') \xrightarrow{f_*} H_n(\mathbf{A}) \xrightarrow{g_*} \cdots$ where $(\partial_*)_{n+1} \colon H_{n+1}(\mathbf{A}') \to H_n(\mathbf{A}'')$ is obtained via the Snake Lemma. 510

25.50, 25.51. Given a map $f \colon M \to N$ of modules, a resolution $\mathbf{A}$ of N, and a projective resolution $\mathbf{P}$ of M, one can lift f to a chain map $\mathbf{f} \colon \mathbf{P} \to \mathbf{A}$ that is unique up to homotopy equivalence. Consequently, any two projective resolutions of a module M are homotopy equivalent. 513, 514

25.54. A right exact covariant functor F is exact iff $L_1F = 0$, in which case $L_nF = 0$ for all n. 515

25.58. The direct sum $\bigoplus M_i$ of right modules is flat iff each M_i is flat. 516

25.59. Every projective module P is flat. 516

25.67 (Shapiro's Lemma). $H_n(G, M_L^G) \cong H_n(L, M)$ for each L-module M and all n; $H^n(G, \operatorname{Coind}_L^G(M)) \cong H^n(L, M)$ for all n. 521

25A.8. An R-module M is a generator in R-**Mod** iff $T(M) = R$. 527

25A.14. If R and R' are Morita equivalent rings, then there is an R-progenerator P such that $R' \cong (\operatorname{End}_R P)\operatorname{op}$. 529

25A.19 (Morita's Theorem). Two rings R, R' are Morita equivalent iff there is an R-progenerator M such that $R' \cong (\operatorname{End}_R M)^{\mathrm{op}}$; in this case the categorical equivalence R-**Mod** $\to R'$-**Mod** is given by $M^* \otimes_R \_$. 531

25A.19′. Notation as in Morita's Theorem, M is also a progenerator in **Mod**-R'. 532

25B.6. The separability idempotent e is indeed an idempotent, and $(r \otimes 1)e = (1 \otimes r)e$ for all $r \in R$. Conversely, if there exists an idempotent $e \in R^e$ satisfying this condition, then R is separable over C, and e is a separability idempotent of R. 533

25B.9. If a module P over a separable C-algebra R is projective as a C-module, then P is projective as an R-module. 534

25B.10. If R is separable over a field F, then R is separable in the classical sense; i.e., R is semisimple and $R \otimes_F \bar{F}$ is semisimple where $\bar{F}$ is the algebraic closure of F. 535

25B.15. If R is separable over its center C, then any maximal ideal B of R has the form AR, where $A = B \cap C \triangleleft C$, and R/AR is central simple over the field C/A. 536

25B.17. Equivalent conditions for a C-algebra R to be Azumaya. 537

25B.20 (Artin-Procesi). A C-algebra R is Azumaya of rank n^2 iff R satisfies all polynomial identities of $M_n(\mathbb{Z})$, and no homomorphic image of R satisfies the standard identity s_{2n-2}. (Other equivalent PI-conditions are also given.) 538

25C.8. Any basic f.d. algebra with $J^2 = 0$ is a homomorphic image of the path algebra $\mathcal{P}(R)$. 543

25C.11 (Gabriel). Suppose R is a f.d. algebra over an algebraically closed field and $J^2 = 0$. Then R has finite representation type iff its quiver (viewed as an undirected graph) is a disjoint union of Dynkin diagrams of types A_n, D_n, E_6, E_7, or E_8. 544

25C.17. Any F-subalgebra R of $M_n(F)$ can be put into block upper triangular form (with respect to a suitable change of base of $F^{(n)}$). 548

E25.6. Every submodule of a projective module over a hereditary ring is projective. 581

E25.7. A fractional ideal P of an integral domain C is invertible (as a fractional ideal) iff P is projective as a module. 581

E25.9, E25.10 (Bourbaki). An example of a module that is invertible and thus projective, but not principal. 581, 582

E25.17. Equivalent conditions for a module over a commutative ring to be invertible. 582

E25B.6. If $H^2(R, \_) = 0$ and R has a nilpotent ideal N such that R/N is separable, then R has a subalgebra $S \cong R/N$ that is a complement to N as a C-module. 591

E25B.11. $0 \to M^R \to M \to \mathrm{Deriv}_C(R, M) \to \mathrm{Ext}^1_{R^e}(R, M) \to 0$ is an exact sequence. 592

E25B.12. Equivalent conditions for an algebra to be separable, in terms of derivations. 593

E25B.14 (Braun). A C-algebra R is Azumaya, iff there are $a_i, b_i \in R$ such that $\sum a_i b_i = 1$ and $\sum a_i R b_i \subseteq C$. 593

E25B.17. Any Azumaya algebra is a finite direct product of algebras of constant rank when the base ring has no nontrivial idempotents. 593

Chapter 26.

26.21 (The Fundamental Theorem of Hopf Modules). Any Hopf module M is isomorphic to $H \otimes M^{\mathrm{co}\,H}$ as Hopf modules (the latter under the "trivial action" $h'(h \otimes a) = (h'h \otimes a)$). 558

26.28 (Nichols-Zoeller [**NiZ**]). If K is a Hopf subalgebra of a f.d. Hopf algebra H, then H is free as a K-module, and $\dim K \mid \dim H$. 561

26.30. A f.d. Hopf algebra H is semisimple iff $\varepsilon(\int_H^l) \neq 0$. 562

E26.3, E26.5. For any algebra A and coalgebra C, $\mathrm{Hom}(C, A)$ becomes an algebra under the convolution product $(*)$. If H is a Hopf algebra, then its antipode S is the inverse to 1_H in $\mathrm{Hom}(H, H)$ under the convolution product. $S(ab) = S(b)S(a)$, $\Delta \circ S = \tau \circ (S \otimes S) \circ \Delta$, and $\epsilon \circ S = \epsilon$. 594

E26.16 (Fundamental Theorem of Comodules). Any finite subset of a comodule M (over a coalgebra C) is contained in a finite-dimensional subcomodule of M. 595

E26.17 (Fundamental Theorem of Coalgebras). Any finite subset of a coalgebra C is contained in a f.d. subcoalgebra of C. 595

E26.28. The following equations hold for $R = \sum a_i \otimes b_i$ in a quasi-triangular Hopf algebra: $R^{-1} = \sum S(a_i) \otimes b_i$; $\sum \epsilon(a_i) b_i = \sum a_i \epsilon(b_i) = 1$; $(S \otimes S)(R) = R$. 596

E26.32. For any almost cocommutative Hopf algebra H with antipode S, there exists invertible $u \in H$ such that $uS(u)$ is central and S^2 is the inner automorphism given by conjugation with respect to u. 597

E26.38. The smash product naturally gives rise to a Morita context $(A\#H, A^H, A, A', \tau, \tau')$. 598

E26.40. The quantum groups of Examples 16A.3 are Hopf algebras. 598

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