

Graduate Algebra: Noncommutative View

Louis Halle Rowen

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in Mathematics**

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To the memory of my beloved mother
Ruth Halle Rowen, April 5, 1918 – January 5, 2007

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Introduction

As indicated in the title, this volume is concerned primarily with noncommutative algebraic structures, having grown from a course introducing complex representations of finite groups via the structure of group algebras and their modules. Our emphasis is on algebras, although we also treat some major classes of finite and infinite groups. Since this volume was conceived as a continuation of Volume 1 (*Graduate Algebra: Commutative View*, Graduate Studies in Mathematics, volume 73), the numeration of chapters starts with Chapter 13, Part IV, and we use the basics of rings and modules developed in Part I of Volume 1 (Chapters 1–3). Nevertheless, Chapters 13–15 and 18 can largely be read independently of Volume 1.

In the last one hundred years there has been a vast literature in noncommutative theory, and our goal here has been to find as much of a common framework as possible. Much of the theory can be cast in terms of representations into matrix algebras, which is our major theme, dominating our treatment of algebras, groups, Lie algebras, and Hopf algebras. A secondary theme is the description of algebraic structures in terms of generators and relations, pursued in the appendices of Chapter 17, and leading to a discussion of free structures, growth, word problems, and Zelmanov’s solution of the Restricted Burnside Problem.

One main divergence of noncommutative theory from commutative theory is that left ideals need not be ideals. Thus, the important notion of “principal ideal” from commutative theory becomes cumbersome; whereas the principal left ideal Ra is described concisely, the smallest ideal of a noncommutative ring QR containing an element a includes all elements of the form

$$r_{1,1}ar_{1,2} + \cdots + r_{m,1}ar_{m,2}, \quad \forall r_{i,1}, r_{i,2} \in R,$$

where m can be arbitrarily large. This forces us to be careful in distinguishing “left” (or “right”) properties from two-sided properties, and leads us to rely heavily on modules.

There are many approaches to structure theory. We have tried to keep our proofs as basic as possible, while at the same time attempting to appeal to a wider audience. Thus, projective modules (Chapter 25) are introduced relatively late in this volume.

The exposition is largely self-contained. Part IV requires basic module theory, especially composition series (Chapter 3 of Volume 1). Chapter 16 draws on material about localization and Noetherian rings from Chapters 8 and 9 of Volume 1. Chapter 17, which goes off in a different direction, requires some material (mostly group theory) given in the prerequisites of this volume. Appendix 17B generalizes the theory of Gröbner bases from Appendix 7B of Volume 1. Chapter 18 has applications to field theory (Chapter 4 of Volume 1).

Parts V and VI occasionally refer to results from Chapters 4, 8, and 10 of Volume 1. At times, we utilize quadratic forms (Appendix 0A) and, occasionally, derivations (Appendix 6B). The end of Chapter 24 draws on material on local fields from Chapter 12. Chapters 25 and 26 require basic concepts from category theory, treated in Appendix 1A.

There is considerable overlap between parts of this volume and my earlier book, *Ring Theory* (student edition), but the philosophy and organization is usually quite different. In *Ring Theory* the emphasis is on the general structure theory of rings, via Jacobson’s Density Theorem, in order to lay the foundations for applications to various kinds of rings.

The course on which this book is based was more goal-oriented — to develop enough of the theory of rings for basic representation theory, i.e., to prove and utilize the Wedderburn-Artin Theorem and Maschke’s Theorem. Accordingly, the emphasis here is on semisimple and Artinian rings, with a short, direct proof. Similarly, the treatment of Noetherian rings here is limited mainly to Goldie’s Theorem, which provides most of the non-technical applications needed later on.

Likewise, whereas in *Ring Theory* we approached representation theory of groups and Lie algebras via ring-theoretic properties of group algebras and enveloping algebras, we focus in Part V of this volume on the actual groups and Lie algebras.

Thanks to Dror Pak for pointing me to the proofs of the hook categories, to Luda Markus-Epstein for material on Stallings foldings, to Alexei Belov for gluing components in the Wedderburn decomposition, and to Sue Montgomery for a description of the current state of the classification of

finite dimensional Hopf algebras. Steve Shnider, Tal Perri, Shai Sarussi, and Luie Plev provided many helpful comments. Again, as with Volume 1, I would like to express special gratitude to David Saltman, in particular for his valuable suggestions concerning Chapter 24 and Chapter 25, and also to Uzi Vishne. Thanks to Sergei Gelfand for having been patient for another two years. And, of course, many thanks to Miriam Beller for much of the technical preparation of the manuscript.

Needless to say, I am deeply indebted to Rachel Rowen, my helpmate, for her steadfast support all of these years.

List of Symbols

Warning: Some notations have multiple meanings.

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Prerequisites

As mentioned in the Introduction, most of Part IV of Volume 2 is self-contained, modulo some basic results on rings and modules. In Chapter 17, we need a few extra general basic results, mostly concerning finitely generated groups, which we list here.

Finitely generated (f.g.) groups.

A fair part of Chapter 17 concerns f.g. groups, introduced briefly in Volume 1, namely on p. 13 and Exercises 0.23–0.27. Often we look for f.g. subgroups of a given f.g. group. The following straightforward facts often come in handy. Recall that a subgroup H has **finite index** G if H has finitely many cosets in G , the number of which is designated as $[G:H]$.

Remark 00.1. Any subgroup H of finite index in a f.g. group G is also f.g. (This was stated in Exercise 0.27 of Volume 1, with an extensive hint.) The same proof shows, more precisely, that if G is generated by t elements and $[G:H] = m$, then H is generated by tm elements.

LEMMA 00.2. *For any $n \in \mathbb{N}$, any f.g. group G has finitely many subgroups of index n .*

Proof. We elaborate on Exercise 0.25 of Volume 1. For any subgroup H of index n , we have a homomorphism $\psi_H: G \rightarrow S_n$, given by left multiplication on the cosets of H . But any element a of $\ker \psi_H$ satisfies $aH = H$, implying $\ker \psi_H \subseteq H$, and thus $H = \psi_H^{-1}(\overline{H})$ for some subgroup \overline{H} of S_n .

Working backwards, since G is f.g., there are only finitely many homomorphisms from G to S_n , which has finitely many possible subgroups \overline{H} . Since any subgroup H of index n can be recovered in this way, we have only finitely many possibilities for H . \square

PROPOSITION 00.3. *If H is a f.g. normal subgroup of G , and K is a subgroup of finite index in H , then K contains a f.g. normal subgroup of G that has finite index in H . (The special case for $H = G$ was given in Exercise 0.25 of Volume 1.)*

Proof. For each $g \in G$, gKg^{-1} is a subgroup of $gHg^{-1} = H$ of the same index as K ; by the lemma, there are only finitely many of these, so, by Exercise 0.24 of Volume 1, $\bigcap_{g \in G} gKg^{-1}$ is a normal subgroup of G having finite index in H . \square

Groups of fractions.

In the proof of Theorem 17.61 we also need the following easy special case of the construction of Exercise 8.26 of Volume 1:

Definition 00.4. Suppose $(A, +)$ is a torsion-free Abelian group. The group $A_{\mathbb{Q}}$ is defined as follows:

Define an equivalence on $A \times \mathbb{N}^+$ by putting $(a, m) \sim (b, n)$, iff $an = bm$. Writing $\frac{a}{m}$ for the equivalence class $[(a, m)]$, we define $A_{\mathbb{Q}}$ to be the set of equivalence classes, endowed with the operation

$$\frac{a}{m} + \frac{b}{n} = \frac{an + bm}{mn}.$$

Remark 00.5. $A_{\mathbb{Q}}$ is a group, and in fact is a \mathbb{Q} -module in the natural way, namely

$$\frac{u}{v} \frac{a}{m} = \frac{ua}{vm}, \quad a \in A, u \in \mathbb{Z}, m, v \in \mathbb{N}^+.$$

There is a group injection $A \rightarrow A_{\mathbb{Q}}$ given by $A \mapsto \frac{a}{1}$. Furthermore, any automorphism σ of A extends naturally to an automorphism of $A_{\mathbb{Q}}$ via the action $\sigma(\frac{a}{m}) = \frac{\sigma(a)}{m}$.

(The verifications are along the lines of those in the proof of Proposition 12.18 of Volume 1. Alternatively, once we have tensor products from Chapter 18, we could view $A_{\mathbb{Q}}$ as $A \otimes_{\mathbb{Z}} \mathbb{Q}$.)

Jordan decomposition.

The Jordan decomposition of Theorem 2.75 of Volume 1 has an easy but useful application in nonzero characteristic:

PROPOSITION 00.6. *Over a field of characteristic $p > 0$, any $n \times n$ matrix T has a power whose radical component is 0.*

Proof. Write the Jordan decomposition $T = T_{\mathbf{s}} + T_{\mathbf{n}}$, where the semisimple component $T_{\mathbf{s}}$ and the nilpotent component $T_{\mathbf{n}}$ commute. Then, as in Corollary 4.69 of Volume 1,

$$T^{p^k} = (T_{\mathbf{s}} + T_{\mathbf{n}})^{p^k} = T_{\mathbf{s}}^{p^k} + T_{\mathbf{n}}^{p^k}$$

for each k , but $T_{\mathbf{n}}^{p^k} = 0$ whenever $p^k > n$, so we conclude for such k that $T^{p^k} = T_{\mathbf{s}}^{p^k}$ is semisimple. \square

Galois theory.

We also need a fact from Galois theory, which was missed in Volume 1.

PROPOSITION 00.7. *Suppose F is a finite field extension of \mathbb{Q} , and $a \in F$ is integral over \mathbb{Z} . If $|\sigma(a)| \leq 1$ for every embedding $\sigma: F \rightarrow \mathbb{C}$, then a is a root of unity.*

Proof. The minimal monic polynomial $f_a \in \mathbb{Z}[\lambda]$ of a over \mathbb{Z} has some degree n ; its coefficients are sums of products of conjugates of a , and so by hypothesis have absolute value $\leq n$. But there are at most $(2n+1)^n$ possibilities for such a polynomial; moreover, the hypothesis also holds for each power of a , which must thus be a root of one of these polynomials. We conclude that there are only finitely many distinct powers of a , which means a is a root of unity. \square

The trace bilinear form.

We need a result about the **trace bilinear form** on the matrix algebra $M_n(F)$ over a field F , given by $\langle x, y \rangle = \text{tr}(xy)$. Clearly this form is symmetric and also nondegenerate, for if $x = (a_{ij})$ with $a_{i_0 j_0} \neq 0$, then $\text{tr}(x e_{j_0 i_0}) = a_{i_0 j_0} \neq 0$. The **discriminant** of a base $\mathcal{B} = \{b_1, \dots, b_{n^2}\}$ of $M_n(F)$ is defined as the determinant of the $n^2 \times n^2$ matrix $(\text{tr}(b_i b_j))$. In view of Remark 4B.5 of Volume 1, the discriminant of any base \mathcal{B} is nonzero (since there exists an orthogonal base with respect to the trace bilinear form).

LEMMA 00.8. *Suppose $\{b_1, \dots, b_n\}$ is a base of $M_n(F)$ over F . Then for any $\alpha_1, \dots, \alpha_n \in F$, the system of n^2 equations $\{\text{tr}(b_i x) = \alpha_i : 1 \leq i \leq n^2\}$ has at most one solution for $x \in M_n(F)$.*

Proof. Write $x = \sum_{j=1}^{n^2} \gamma_j b_j$. Then $\alpha_i = \sum_{j=1}^{n^2} \gamma_j \text{tr}(b_i b_j)$, $1 \leq i \leq n^2$, can be viewed as n^2 equations in the γ_j ; since the discriminant $\det(\text{tr}(b_i b_j))$ is nonzero, one can solve these equations using Cramer's rule.

To prove uniqueness, suppose there were two matrices x_1 and x_2 such that $\text{tr}(b_i x_1) = \text{tr}(b_i x_2)$, $1 \leq i \leq n^2$. Then $\text{tr}(b_i(x_1 - x_2)) = 0$ for each i , which implies $x_1 - x_2 = 0$ since the trace form is nondegenerate; thus, $x_1 = x_2$. \square

List of Major Results

The prefix E denotes that the referred result is an exercise, such as E0.5. Since the exercises do not necessarily follow a chapter immediately, their page numbers may be out of sequence.

Prerequisites.

00.3. Any subgroup of finite index in a f.g. normal subgroup H contains a f.g. normal subgroup of G of finite index in H xxiv

00.6. Any $n \times n$ matrix has a power that is semisimple. xxiv

00.7. If $a \in F$ is integral over \mathbb{Z} and $|\sigma(a)| \leq 1$ for every embedding $\sigma: F \rightarrow \mathbb{C}$, then a is a root of unity. xxv

Chapter 13.

13.9. Any ring W having a set of $n \times n$ matrix units has the form $M_n(R)$, where $R = e_{11}We_{11}$. 10

13.14. There is a lattice isomorphism $\{\text{Ideals of } R\} \rightarrow \{\text{Ideals of } M_n(R)\}$ given by $A \mapsto M_n(A)$. 13

13.18. For any division ring D , the ring $M_n(D)$ is a direct sum of n minimal left ideals, and thus has composition length n . 14

13.31. Determination of the modules, left ideals, and ideals for a finite direct product of rings. 18

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$$\begin{array}{ccccccc} A_1'' & \xrightarrow{f_1} & A_1 & \xrightarrow{g_1} & A_1' & \longrightarrow & 0 \\ d'' \downarrow & & d \downarrow & & d' \downarrow & & \\ 0 & \longrightarrow & A_2'' & \xrightarrow{f_2} & A_2 & \xrightarrow{g_2} & A_2' \end{array}$$

gives rise to an exact sequence $\ker d'' \rightarrow \ker d \rightarrow \ker d' \rightarrow \operatorname{coker} d'' \rightarrow \operatorname{coker} d \rightarrow \operatorname{coker} d'$. 506

25.44. For any exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ of modules and respective projective resolutions (\mathbb{P}', d') and (\mathbb{P}'', d'') of M' and M'' , there exists a projective resolution (\mathbb{P}, d) of M , such that $P_n = P_n' \oplus P_n''$ for each n , and the three projective resolutions form a commutative diagram. 509

25.45. Any short exact sequence $0 \rightarrow (\mathbf{A}'', d'') \xrightarrow{f} (\mathbf{A}, d) \xrightarrow{g} (\mathbf{A}', d') \rightarrow 0$ of complexes gives rise to a long exact sequence of the homology groups $\cdots \rightarrow H_{n+1}(\mathbf{A}'') \xrightarrow{f_*} H_{n+1}(\mathbf{A}) \xrightarrow{g_*} H_{n+1}(\mathbf{A}') \xrightarrow{\partial_*} H_n(\mathbf{A}'') \xrightarrow{f_*} H_n(\mathbf{A}) \xrightarrow{g_*} \cdots$ where $(\partial_*)_{n+1}: H_{n+1}(\mathbf{A}') \rightarrow H_n(\mathbf{A}'')$ is obtained via the Snake Lemma. 510

25.50, 25.51. Given a map $f: M \rightarrow N$ of modules, a resolution \mathbf{A} of N , and a projective resolution \mathbf{P} of M , one can lift f to a chain map $\mathbf{f}: \mathbf{P} \rightarrow \mathbf{A}$ that is unique up to homotopy equivalence. Consequently, any two projective resolutions of a module M are homotopy equivalent. 513, 514

25.54. A right exact covariant functor F is exact iff $L_1 F = 0$, in which case $L_n F = 0$ for all n . 515

25.58. The direct sum $\bigoplus M_i$ of right modules is flat iff each M_i is flat. 516

25.59. Every projective module P is flat. 516

25.67 (Shapiro's Lemma). $H_n(G, M_L^G) \cong H_n(L, M)$ for each L -module M and all n ; $H^n(G, \operatorname{Coind}_L^G(M)) \cong H^n(L, M)$ for all n . 521

25A.8. An R -module M is a generator in $R\text{-Mod}$ iff $T(M) = R$. 527

25A.14. If R and R' are Morita equivalent rings, then there is an R -progenerator P such that $R' \cong (\operatorname{End}_R P) \operatorname{op}$. 529

25A.19 (Morita's Theorem). Two rings R, R' are Morita equivalent iff there is an R -progenerator M such that $R' \cong (\operatorname{End}_R M)^{\operatorname{op}}$; in this case the categorical equivalence $R\text{-Mod} \rightarrow R'\text{-Mod}$ is given by $M^* \otimes_R _$. 531

25A.19'. Notation as in Morita's Theorem, M is also a progenerator in $\mathbf{Mod}\text{-}R'$. 532

25B.6. The separability idempotent e is indeed an idempotent, and $(r \otimes 1)e = (1 \otimes r)e$ for all $r \in R$. Conversely, if there exists an idempotent $e \in R^e$ satisfying this condition, then R is separable over C , and e is a separability idempotent of R . 533

25B.9. If a module P over a separable C -algebra R is projective as a C -module, then P is projective as an R -module. 534

25B.10. If R is separable over a field F , then R is separable in the classical sense; i.e., R is semisimple and $R \otimes_F \bar{F}$ is semisimple where \bar{F} is the algebraic closure of F . 535

25B.15. If R is separable over its center C , then any maximal ideal B of R has the form AR , where $A = B \cap C \triangleleft C$, and R/AR is central simple over the field C/A . 536

25B.17. Equivalent conditions for a C -algebra R to be Azumaya. 537

25B.20 (Artin-Procesi). A C -algebra R is Azumaya of rank n^2 iff R satisfies all polynomial identities of $M_n(\mathbb{Z})$, and no homomorphic image of R satisfies the standard identity s_{2n-2} . (Other equivalent PI-conditions are also given.) 538

25C.8. Any basic f.d. algebra with $J^2 = 0$ is a homomorphic image of the path algebra $\mathcal{P}(R)$. 543

25C.11 (Gabriel). Suppose R is a f.d. algebra over an algebraically closed field and $J^2 = 0$. Then R has finite representation type iff its quiver (viewed as an undirected graph) is a disjoint union of Dynkin diagrams of types A_n, D_n, E_6, E_7 , or E_8 . 544

25C.17. Any F -subalgebra R of $M_n(F)$ can be put into block upper triangular form (with respect to a suitable change of base of $F^{(n)}$). 548

E25.6. Every submodule of a projective module over a hereditary ring is projective. 581

E25.7. A fractional ideal P of an integral domain C is invertible (as a fractional ideal) iff P is projective as a module. 581

E25.9, E25.10 (Bourbaki). An example of a module that is invertible and thus projective, but not principal. 581, 582

E25.17. Equivalent conditions for a module over a commutative ring to be invertible. 582

- E25.20 (Schanuel's Lemma). If $0 \rightarrow K_i \rightarrow P_i \rightarrow M \rightarrow 0$ are exact with P_i projective for $i = 1, 2$, then $P_1 \oplus K_2 \cong P_2 \oplus K_1$. 582
- E25.22, E25.23. Inequalities involving projective dimensions of modules in an exact sequence. 583
- E25.24. $\text{pd}_{R[\lambda]} M \leq \text{pd}_R M + 1$ for any $R[\lambda]$ -module M . 583
- E25.25. (Eilenberg). For any projective module P , the module $P \oplus F$ is free for some free module F . 583
- E25.28. (Baer's criterion). To verify injectivity, it is enough to check Equation (25.5) for $M = R$. 584
- E25.37. $P^* = \text{Hom}_C(P, E)$ is injective, for any flat right R -module P and any injective C -module E . 584
- E25.39. Any module has an injective hull. 585
- E25.45. For any adjoint pair (F, G) of functors, F is right exact and G is left exact. 585
- E25.47. Any homological δ -functor defined by a bifunctor is independent of the choice of component. 586
- E25.53. The homology functor is a universal δ -functor. 587
- E25.55 (Generic flatness). If $S^{-1}M$ is free as an $S^{-1}C$ -module, then there is $s \in S$ such that $M[s^{-1}]$ is free as a $C[s^{-1}]$ -module. 587
- E25.56. Every finitely presented flat module is projective. 587
- E25.58. Group algebras over a field are quasi-Frobenius. 587
- E25.61. $\text{gl dim } R = \sup\{n : \text{Ext}^n(M, N) \neq 0 \text{ for all } R\text{-modules } M, N\} = \sup\{\text{injective dimensions of all } R\text{-modules}\}$. 587
- E25.65. $\text{Ext}^1(M, N)$ can be identified with the equivalence classes of module extensions $0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0$. 588
- E25.69. The corestriction map is compatible with the transfer in the cohomology of $H^2(G, K^\times)$. 589
- E25.71. $H^1(L, M) = \text{Deriv}(L)/\text{InnDeriv}(L)$ for any Lie algebra L . 589
- E25A.6. Morita equivalent commutative rings are isomorphic. 590
- E25A.9. Properties of Morita contexts with τ, τ' onto. 590

E25B.6. If $H^2(R, _) = 0$ and R has a nilpotent ideal N such that R/N is separable, then R has a subalgebra $S \cong R/N$ that is a complement to N as a C -module. 591

E25B.11. $0 \rightarrow M^R \rightarrow M \rightarrow \text{Deriv}_C(R, M) \rightarrow \text{Ext}_{R^e}^1(R, M) \rightarrow 0$ is an exact sequence. 592

E25B.12. Equivalent conditions for an algebra to be separable, in terms of derivations. 593

E25B.14 (Braun). A C -algebra R is Azumaya, iff there are $a_i, b_i \in R$ such that $\sum a_i b_i = 1$ and $\sum a_i R b_i \subseteq C$. 593

E25B.17. Any Azumaya algebra is a finite direct product of algebras of constant rank when the base ring has no nontrivial idempotents. 593

Chapter 26.

26.21 (The Fundamental Theorem of Hopf Modules). Any Hopf module M is isomorphic to $H \otimes M^{\text{co}H}$ as Hopf modules (the latter under the “trivial action” $h'(h \otimes a) = (h'h \otimes a)$). 558

26.28 (Nichols-Zoeller [NiZ]). If K is a Hopf subalgebra of a f.d. Hopf algebra H , then H is free as a K -module, and $\dim K \mid \dim H$. 561

26.30. A f.d. Hopf algebra H is semisimple iff $\varepsilon(\int_H^l) \neq 0$. 562

E26.3, E26.5. For any algebra A and coalgebra C , $\text{Hom}(C, A)$ becomes an algebra under the convolution product $(*)$. If H is a Hopf algebra, then its antipode S is the inverse to 1_H in $\text{Hom}(H, H)$ under the convolution product. $S(ab) = S(b)S(a)$, $\Delta \circ S = \tau \circ (S \otimes S) \circ \Delta$, and $\varepsilon \circ S = \varepsilon$. 594

E26.16 (Fundamental Theorem of Comodules). Any finite subset of a comodule M (over a coalgebra C) is contained in a finite-dimensional subcomodule of M . 595

E26.17 (Fundamental Theorem of Coalgebras). Any finite subset of a coalgebra C is contained in a f.d. subcoalgebra of C . 595

E26.28. The following equations hold for $R = \sum a_i \otimes b_i$ in a quasi-triangular Hopf algebra: $R^{-1} = \sum S(a_i) \otimes b_i$; $\sum \varepsilon(a_i) b_i = \sum a_i \varepsilon(b_i) = 1$; $(S \otimes S)(R) = R$. 596

E26.32. For any almost cocommutative Hopf algebra H with antipode S , there exists invertible $u \in H$ such that $uS(u)$ is central and S^2 is the inner automorphism given by conjugation with respect to u . 597

E26.38. The smash product naturally gives rise to a Morita context $(A\#H, A^H, A, A', \tau, \tau')$. 598

E26.40. The quantum groups of Examples 16A.3 are Hopf algebras. 598

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