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# Contents

# Preface

In this book, we study the evolution of Riemannian metrics under the Ricci flow. This evolution equation was introduced in a seminal paper by R. Hamilton [**44**], following earlier work of Eells and Sampson [**33**] on the harmonic map heat flow. Using the Ricci flow, Hamilton proved that every compact three-manifold with positive Ricci curvature is diffeomorphic to a spherical space form. The Ricci flow has since been used to resolve longstanding open questions in Riemannian geometry and three-dimensional topology. In this text, we focus on the convergence theory for the Ricci flow in higher dimensions and its application to the Differentiable Sphere Theorem. The results we describe have all appeared in research articles. However, we have made an effort to simplify various arguments and streamline the exposition.

In Chapter 1, we give a survey of various sphere theorems in Riemannian geometry (see also [**22**]). We first describe the Topological Sphere Theorem of Berger and Klingenberg. We then discuss various generalizations of that theorem, such as the Diameter Sphere Theorem of Grove and Shiohama [**42**] and the Sphere Theorem of Micallef and Moore [**60**]. These results rely on the variational theory for geodesics and harmonic maps, respectively. We will discuss the main ideas involved in the proof; however, this material will not be used in later chapters. Finally, we state the Differentiable Sphere Theorem obtained by the author and R. Schoen [**20**].

In Chapter 2, we state the definition of the Ricci flow and describe the short-time existence and uniqueness theory. We then study how the Riemann curvature tensor changes when the metric evolves under the Ricci flow. This evolution equation will be the basis for all the *a priori* estimates established in later chapters.

In Chapter 3, we describe Shi's estimates for the covariant derivatives of the curvature tensor. As an application, we show that the Ricci flow cannot develop a singularity in finite time unless the curvature is unbounded. Moreover, we establish interior estimates for solutions of linear parabolic equations. These estimates play an important role in Sections 4.3 and 5.4.

In Chapter 4, we consider the Ricci flow on $S^2$. In Section 4.1, we show that any gradient Ricci soliton on $S^2$ has constant curvature. We then study solutions to the Ricci flow on $S^2$ with positive scalar curvature. A theorem of Hamilton [**46**] asserts that such a solution converges to a constant curvature metric after rescaling. A key ingredient in the proof is the monotonicity of Hamilton's entropy functional. This monotonicity formula will be discussed in Section 4.2. Alternative proofs of this theorem can be found in [**4**], [**6**], [**48**], or [**82**]. The arguments in [**4**] and [**48**] are based on a careful study of the isoperimetric profile, while the proofs in [**6**] and [**82**] employ PDE techniques.

In Chapter 5, we describe Hamilton's maximum principle for the Ricci flow and discuss the notion of a pinching set. We then describe a general convergence criterion for the Ricci flow. This criterion was discovered by Hamilton [**45**] and plays an important role in the study of Ricci flow.

In Chapter 6, we explain how Hamilton's classification of three-manifolds with positive Ricci curvature follows from the general theory developed in Chapter 5. We then describe an important curvature estimate, due to Hamilton and Ivey. This inequality holds for any solution to the Ricci flow in dimension 3.

In Chapter 7, we describe various curvature conditions which are preserved by the Ricci flow. We first prove that nonnegative isotropic curvature is preserved by the Ricci flow in all dimensions. This curvature condition originated in Micallef and Moore's work on the Morse index of harmonic two-spheres and plays a central role in this book. We then consider the condition that $M \times \mathbb{R}$ has nonnegative isotropic curvature. This condition is stronger than nonnegative isotropic curvature, and is also preserved by the Ricci flow. Continuing in this fashion, we consider the condition that $M \times \mathbb{R}^2$ has nonnegative isotropic curvature, and the condition that $M \times S^2(1)$ has nonnegative isotropic curvature. (Here, $S^2(1)$ denotes a two-dimensional sphere of constant curvature 1.) We show that these conditions are preserved by the Ricci flow as well.

In Chapter 8, we present the proof of the Differentiable Sphere Theorem. More generally, we show that every compact Riemannian manifold $M$ with the property that $M \times \mathbb{R}$ has positive isotropic curvature is diffeomorphic to a spherical space form. This theorem is the main result of Chapter 8. It can be viewed as a generalization of Hamilton's work in dimension 3 and was originally proved in [**17**].

In Chapter 9, we prove various rigidity theorems. In particular, we classify all compact Riemannian manifolds $M$ with the property that $M \times \mathbb{R}$ has nonnegative isotropic curvature. Moreover, we show that any Einstein manifold with nonnegative isotropic curvature is necessarily locally symmetric. This generalizes classical results due to Berger [**10**], [**11**] and Tachibana [**84**]. In order to handle the borderline case, we employ a variant of Bony's strict maximum principle for degenerate elliptic equations.

The material presented in Chapters 2–9 is largely, though not fully, self-contained. In Section 2.2, we employ the existence and uniqueness theory for parabolic systems. In Section 4.2, we use the convergence theory for Riemannian manifolds developed by Cheeger and Gromov. Finally, in Chapter 9, we use Berger's classification of holonomy groups, as well as some basic facts about Kähler and quaternionic-Kähler manifolds.

There are some important aspects of Ricci flow which are not mentioned in this book. For example, we do not discuss Hamilton's differential Harnack inequality (cf. [**47**], [**49**]) or Perelman's crucial monotonicity formulae (see [**68**], [**69**]). A detailed exposition of Perelman's entropy functional can be found in [**63**] or [**85**]. A generalization of Hamilton's Harnack inequality is described in [**18**] (see also [**24**]).

This book grew out of a *Nachdiplom* course given at ETH Zürich. It is a pleasure to thank the Department of Mathematics at ETH Zürich for its hospitality. I am especially grateful to Professor Michael Struwe and Professor Tristan Rivière for many inspiring discussions. Without their encouragement, this book would never have been written. Finally, I thank Professor Camillo De Lellis for valuable comments on an earlier version of this manuscript.

# Bibliography


[1] D. Alekseevskii, *Riemannian spaces with exceptional holonomy groups,* Functional Anal. Appl. 2, 97–105 (1968)

[2] M. Anderson, *Ricci curvature bounds and Einstein metrics on compact manifolds,* J. Amer. Math. Soc. 2, 455–490 (1989)

[3] B. Andrews, *Contraction of convex hypersurfaces in Riemannian spaces,* J. Diff. Geom. 39, 407–431 (1994)

[4] B. Andrews and P. Bryan, *Curvature bounds by isoperimetric comparison for normalized Ricci flow on the two-sphere,* arxiv:0908.3606

[5] B. Andrews and H. Nguyen, *Four-manifolds with* 1/4*-pinched flag curvatures,* Asian J. Math. 13, 251–270 (2009)

[6] J. Bartz, M. Struwe, and R. Ye, *A new approach to the Ricci flow on* $S^2$*,* Ann. Scuola Norm. Sup. Pisa Serie IV, 21, 475–482 (1994)

[7] M. Berger, *Sur les groupes d'holonomie homogènes des variétés à connexion affine et des variétés riemanniennes,* Bull. Soc. Math. France 283, 279–330 (1955)

[8] M. Berger, *Les variétés Riemanniennes* 1/4*-pincées,* Ann. Scuola Norm. Sup. Pisa Serie III, 14, 161–170 (1960)

[9] M. Berger, *Sur quelques variétés riemanniennes suffisamment pincées,* Bull. Soc. Math. France 88, 57–71 (1960)

[10] M. Berger, *Sur quelques variétés d'Einstein compactes,* Ann. Mat. Pura Appl. 53, 89–95 (1961)

[11] M. Berger, *Sur les variétés d'Einstein compactes,* Comptes Rendus de la IIIe Réunion du Groupement des Mathématiciens d'Expression Latine (Namur, 1965), 35–55, Librairie Universitaire, Louvain (1966)

[12] M. Berger, *Trois remarques sur les variétés riemanniennes à courbure positive,* C. R. Acad. Sci. Paris Sér. A-B 263, A76–A78 (1966)

[13] A. Besse, *Einstein manifolds,* Classics in Mathematics, Springer-Verlag, Berlin (2008)

[14] C. Böhm and B. Wilking, *Manifolds with positive curvature operator are space forms,* Ann. of Math. (2) 167, 1079–1097 (2008)

[15] J.M. Bony, *Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés,* Ann. Inst. Fourier (Grenoble) 19, 277–304 (1969)

[16] G.E. Bredon, *Topology and Geometry,* Graduate Texts in Mathematics, vol. 139, Springer-Verlag, New York (1993)

[17] S. Brendle, *A general convergence result for the Ricci flow,* Duke Math. J. 145, 585–601 (2008)

[18] S. Brendle, *A generalization of Hamilton's differential Harnack inequality for the Ricci flow,* J. Diff. Geom. 82, 207–227 (2009)

[19] S. Brendle, *Einstein manifolds with nonnegative isotropic curvature are locally symmetric,* Duke Math. J. 151, 1–21 (2010)

[20] S. Brendle and R. Schoen, *Manifolds with* 1/4*-pinched curvature are space forms,* J. Amer. Math. Soc. 22, 287–307 (2009)

[21] S. Brendle and R. Schoen, *Classification of manifolds with weakly* 1/4*-pinched curvatures,* Acta Math. 200, 1–13 (2008)

[22] S. Brendle and R. Schoen, *Sphere theorems in geometry,* Surveys in Differential Geometry, vol. XIII, 49–84, International Press, Somerville MA (2009)

[23] R.B. Brown and A. Gray, *Riemannian manifolds with holonomy group* Spin(9)*,* Differential Geometry (in Honor of K. Yano), 41–59, Kinokuniya, Tokyo (1972)

[24] H.D. Cao, *On Harnack's inequalities for the Kähler-Ricci flow,* Invent. Math. 109, 247–263 (1992)

[25] A. Chang, M. Gursky, and P. Yang, *A conformally invariant sphere theorem in four dimensions,* Publ. Math. IHÉS 98, 105–143 (2003)

[26] J. Cheeger and D. Ebin, *Comparison theorems in Riemannian geometry,* AMS Chelsea Publishing, Providence RI (2008)

[27] J. Cheeger and D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature,* J. Diff. Geom. 6, 119–128 (1971)

[28] H. Chen, *Pointwise* 1/4*-pinched* 4*-manifolds,* Ann. Global Anal. Geom. 9, 161–176 (1991)

[29] X. Chen, P. Lu, and G. Tian, *A note on uniformization of Riemann surfaces by Ricci flow,* Proc. Amer. Math. Soc. 134, 3391–3393 (2006)

[30] B. Chow, *The Ricci flow on the* 2*-sphere,* J. Diff. Geom. 33, 325–334 (1991)

[31] B. Chow and L.F. Wu, *The Ricci flow on compact* 2*-orbifolds with curvature negative somewhere,* Comm. Pure Appl. Math. 44, 275–286 (1991)

[32] D. DeTurck, *Deforming metrics in the direction of their Ricci tensors,* J. Diff. Geom. 18, 157–162 (1983)

[33] J. Eells, Jr., and J.H. Sampson, *Harmonic mappings of Riemannian manifolds,* Amer. J. Math. 86, 109–160 (1964)

[34] J.-H. Eschenburg, *Local convexity and nonnegative curvature – Gromov's proof of the sphere theorem,* Invent. Math. 84, 507–522 (1986)

[35] A. Fraser, *Fundamental groups of manifolds with positive isotropic curvature,* Ann. of Math. (2) 158, 345–354 (2003)

[36] M.H. Freedman, *The topology of four-dimensional manifolds,* J. Diff. Geom. 17, 357–453 (1982)

[37] S. Goldberg and S. Kobayashi, *Holomorphic bisectional curvature,* J. Diff. Geom. 1, 225–233 (1967)

[38] D. Gromoll, *Differenzierbare Strukturen und Metriken positiver Krümmung auf Sphären,* Math. Ann. 164, 353–371 (1966)

[39] M. Gromov, *Positive curvature, macroscopic dimension, spectral gaps and higher signatures,* Functional Analysis on the Eve of the 21st Century, vol. II (New Brunswick 1993), 1–213, Progr. Math., 132, Birkhäuser, Boston (1996)

[40] A. Grothendieck, *Sur la classification des fibrés holomorphes sur la sphère de Riemann,* Amer. J. Math. 79, 121–138 (1957)

[41] K. Grove, H. Karcher, and E. Ruh, *Jacobi fields and Finsler metrics on compact Lie groups with an application to differentiable pinching problems,* Math. Ann. 211, 7–21 (1974)

[42] K. Grove and K. Shiohama, *A generalized sphere theorem,* Ann. of Math. (2) 106, 201–211 (1977)

[43] M. Gursky and C. LeBrun, *On Einstein manifolds of positive sectional curvature,* Ann. Global Anal. Geom. 17, 315–328 (1999)

[44] R. Hamilton, *Three-manifolds with positive Ricci curvature,* J. Diff. Geom. 17, 255–306 (1982)

[45] R. Hamilton, *Four-manifolds with positive curvature operator,* J. Diff. Geom. 24, 153–179 (1986)

[46] R. Hamilton, *The Ricci flow on surfaces,* Contemp. Math. 71, 237–262, Amer. Math. Soc., Providence RI (1988)

[47] R. Hamilton, *The Harnack estimate for the Ricci flow,* J. Diff. Geom. 37, 225–243 (1993)

[48] R. Hamilton, *An isoperimetric estimate for the Ricci flow on the two-sphere,* Modern Methods in Complex Analysis (Princeton 1992), 191–200, Ann. of Math. Stud. 137, Princeton University Press, Princeton NJ (1995)

[49] R. Hamilton, *The formation of singularities in the Ricci flow,* Surveys in Differential Geometry, vol. II, 7–136, International Press, Somerville MA (1995)

[50] R. Hamilton, Lectures given at Harvard University (1996)

[51] R. Hamilton, *Four-manifolds with positive isotropic curvature,* Comm. Anal. Geom. 5, 1–92 (1997)

[52] R. Hamilton, *Three-orbifolds with positive Ricci curvature,* Collected Papers on Ricci flow, 521–524, Ser. Geom. Topol. 37, International Press, Somerville MA (2003)

[53] S. Helgason, *Differential geometry and symmetric spaces,* Academic Press, New York (1962)

[54] G. Huisken, *Ricci deformation of the metric on a Riemannian manifold,* J. Diff. Geom. 21, 47–62 (1985)

[55] T. Ivey, *Ricci solitons on compact three-manifolds,* Diff. Geom. Appl. 3, 301–307 (1993)

[56] W. Klingenberg, *Über Riemannsche Mannigfaltigkeiten mit positiver Krümmung,* Comment. Math. Helv. 35, 47–54 (1961)

[57] C. Margerin, *Pointwise pinched manifolds are space forms,* Geometric Measure Theory and the Calculus of Variations (Arcata 1984), 343–352, Proc. Sympos. Pure Math. 44, Amer. Math. Soc., Providence RI (1986)

[58] C. Margerin, *A sharp characterization of the smooth* 4*-sphere in curvature terms,* Comm. Anal. Geom. 6, 21–65 (1998)

[59] D. Meyer, *Sur les variétés riemanniennes à opérateur de courbure positif,* C. R. Acad. Sci. Paris Sér. A-B 272, A482–A485 (1971)

[60] M. Micallef and J.D. Moore, *Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes,* Ann. of Math. (2) 127, 199-227 (1988)

[61] M. Micallef and M. Wang, *Metrics with nonnegative isotropic curvature,* Duke Math. J. 72, 649–672 (1993)

[62] J. Milnor, *On manifolds homeomorphic to the* 7*-sphere,* Ann. of Math. (2) 64, 399–405 (1956)

[63] R. Müller, *Differential Harnack inequalities and the Ricci flow,* European Mathematical Society, Zürich (2006)

[64] H. Nguyen, *Isotropic curvature and the Ricci flow,* Internat. Math. Res. Notices (to appear)

[65] S. Nishikawa, *Deformation of Riemannian metrics and manifolds with bounded curvature ratios,* Geometric Measure Theory and the Calculus of Variations (Arcata 1984), 343–352, Proc. Sympos. Pure Math. 44, Amer. Math. Soc., Providence RI (1986)

[66] C. Olmos, *A geometric proof of the Berger holonomy theorem,* Ann. of Math. (2) 161, 579–588 (2005)

[67] B. Osgood, R. Phillips, and P. Sarnak, *Extremals of determinants of Laplacians,* J. Funct. Anal. 80, 148–211 (1988)

[68] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications,* arxiv:0211159

[69] G. Perelman, *Ricci flow with surgery on three-manifolds,* arxiv:0303109

[70] P. Petersen and T. Tao, *Classification of almost quarter-pinched manifolds,* Proc. Amer. Math. Soc. 137, 2437–2440 (2009)

[71] H.E. Rauch, *A contribution to differential geometry in the large,* Ann. of Math. (2) 54, 38–55 (1951)

[72] E. Ruh, *Krümmung und differenzierbare Struktur auf Sphären II,* Math. Ann. 205, 113–129 (1973)

[73] E. Ruh, *Riemannian manifolds with bounded curvature ratios,* J. Diff. Geom. 17, 643–653 (1982)

[74] S. Salamon, *Quaternionic Kähler manifolds,* Invent. Math. 67, 143–171 (1982)

[75] R. Schoen and S.T. Yau, *Lectures on Harmonic Maps,* International Press, Cambridge (1997)

[76] H. Seshadri, *Manifolds with nonnegative isotropic curvature,* Comm. Anal. Geom. (to appear)

[77] N. Šešum, *Curvature tensor under the Ricci flow,* Amer. J. Math. 127, 1315–1324 (2005)

[78] W.X. Shi, *Deforming the metric on complete Riemannian manifolds,* J. Diff. Geom. 30, 223–301 (1989)

[79] J. Simons, *On the transitivity of holonomy systems,* Ann. of Math. (2) 76, 213–234 (1962)

[80] Y.T. Siu and S.T. Yau, *Compact Kähler manifolds of positive bisectional curvature,* Invent. Math. 59, 189–204 (1980)

[81] S. Smale, *Generalized Poincaré's conjecture in dimensions greater than four,* Ann. of Math. (2) 74, 391–406 (1961)

[82] M. Struwe, *Curvature flows on surfaces,* Ann. Scuola Norm. Sup. Pisa Serie V, 1, 247–274 (2002)

[83] M. Sugimoto, K. Shiohama, and H. Karcher, *On the differentiable pinching problem,* Math. Ann. 195, 1–16 (1971)

[84] S. Tachibana, *A theorem on Riemannian manifolds with positive curvature operator,* Proc. Japan Acad. 50, 301–302 (1974)

[85] P. Topping, *Lectures on the Ricci Flow,* London Mathematical Society Lecture Note Series, vol. 325, Cambridge University Press, Cambridge (2006)

[86] L.F. Wu, *The Ricci flow on* 2*-orbifolds with positive curvature,* J. Diff. Geom. 33, 575–596 (1991)

[87] D. Yang, *Rigidity of Einstein* 4*-manifolds with positive curvature,* Invent. Math. 142, 435–450 (2000)

# Index

In 1982, R. Hamilton introduced a nonlinear evolution equation for Riemannian metrics with the aim of finding canonical metrics on manifolds. This evolution equation is known as the Ricci flow, and it has since been used widely and with great success, most notably in Perelman's solution of the Poincaré conjecture. Furthermore, various convergence theorems have been established.

This book provides a concise introduction to the subject as well as a comprehensive account of the convergence theory for the Ricci flow. The proofs rely mostly on maximum principle arguments. Special emphasis is placed on preserved curvature conditions, such as positive isotropic curvature. One of the major consequences of this theory is the Differentiable Sphere Theorem: a compact Riemannian manifold, whose sectional curvatures all lie in the interval (1,4], is diffeomorphic to a spherical space form. This question has a long history, dating back to a seminal paper by H. E. Rauch in 1951, and it was resolved in 2007 by the author and Richard Schoen.

This text originated from graduate courses given at ETH Zürich and Stanford University, and it is directed at graduate students and researchers. The reader is assumed to be familiar with basic Riemannian geometry, but no previous knowledge of Ricci flow is required.

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