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To the memory of Jerald Joseph Kovacic (1941–2009).

Contents

Preface

The aim of this book is to present the Galois theory of homogeneous linear differential equations. This theory goes back to the work of Picard and Vessiot at the end of the 19th century and bears their names. It parallels the Galois theory of algebraic equations. The notions of splitting field, Galois group, and solvability by radicals have their counterparts in the notions of Picard-Vessiot extension, differential Galois group, and solvability by quadratures. The differential Galois group of a homogeneous linear differential equation has a structure of linear algebraic group; hence it is endowed, in particular, with the Zariski topology. The fundamental theorem of Picard-Vessiot theory establishes a bijective correspondence between intermediate differential fields of a Picard-Vessiot extension and closed subgroups of its differential Galois group. Solvability by quadratures is characterized by means of the differential Galois group. Picard-Vessiot theory was clarified and generalized in the work of Kolchin in the mid-20th century. Kolchin used the differential algebra developed by Ritt and also built the foundations of the theory of linear algebraic groups. Kaplansky's book "Introduction to Differential Algebra" made the theory more accessible, although it omits an important point, namely the construction of the Picard-Vessiot extension. The more recent books by Magid and van der Put and Singer assume that the reader is familiar with algebraic varieties and linear algebraic groups, although the latter book compiles the most important topics in an appendix. We point out that not all results on algebraic varieties and algebraic groups needed to develop differential Galois theory appear in the standard books on these topics. For our book we have decided to develop the theory of algebraic varieties and linear algebraic groups in the same way that books on classical Galois theory include some chapters on group, ring, and field theories. Our text includes complete proofs, both of the results on algebraic geometry and algebraic groups which are needed in Picard-Vessiot theory and of the results on Picard-Vessiot theory itself.

We have given several courses on Differential Galois Theory in Barcelona and Kraków. As a result, we published our previous book "Introduction to Differential Galois Theory" [**C-H1**]. Although published by a university publishing house, it has made some impact and has been useful to graduate students as well as to theoretical physicists working on dynamical systems. Our present book is also aimed at graduate students in mathematics or physics and at researchers in these fields looking for an introduction to the subject. We think it is suitable for a graduate course of one or two semesters, depending on students' backgrounds in algebraic geometry and algebraic groups. Interested students can work out the exercises, some of which give an insight into topics beyond the ones treated in this book. The prerequisites for this book are undergraduate courses in commutative algebra and complex analysis.

We would like to thank our colleagues José María Giral, Andrzej Nowicki, and Henryk Żołądek who carefully read parts of this book and made valuable comments, as well as Jakub Byszewski and Sławomir Cynk for interesting discussions on its content. We are also grateful to the anonymous referees for their corrections and suggestions which led to improvements in the text. Our thanks also go to Dr. Ina Mette for persuading us to expand our previous book to create the present one and for her interest in this project.

Finally our book owes much to Jerry Kovacic. We will always be thankful to him for many interesting discussions and will remember him as a brilliant mathematician and an open and friendly person.

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Barcelona and Kraków, October 2010

Teresa Crespo and Zbigniew Hajto

Introduction

This book has been conceived as a self-contained introduction to differential Galois theory. The self-teaching reader or the teacher wanting to give a course on this subject will find complete proofs of all results included. We have chosen to make a classical presentation of the theory. We refer to the Picard-Vessiot extension as a field rather than introducing the notion of Picard-Vessiot ring so as to keep the analogy with the splitting field in the polynomial Galois theory. We also refer to differential equations rather than differential systems, although the differential systems setting is given in the exercises.

The chapters on algebraic geometry and algebraic groups include all questions which are necessary to develop differential Galois theory. The differential Galois group of a linear differential equation is a linear algebraic group, hence affine. However, the construction of the quotient of an algebraic group by a subgroup needs the notion of abstract affine variety. Once we introduce the notion of geometric space, the concept of algebraic variety comes naturally. We also consider it interesting to include the notion of projective variety, which is a model for algebraic varieties, and present a classical example of an algebraic group which is not affine, namely the elliptic curve.

The chapter on Lie algebras aims to prove the equivalence between the solvability of a connected linear algebraic group and the solvability of its Lie algebra. This fact is used in particular to determine the algebraic subgroups of $\mathrm{SL}(2,\mathbb{C})$. We present the characterization of differential equations solvable by quadratures. In the last chapter we consider differential equations defined over the field of rational functions over the complex field and present classical notions such as the monodromy group, Fuchsian equations and hypergeometric equations. The last section is devoted to Kovacic's powerful algorithm to compute Liouvillian solutions to linear differential equations of order 2. Each chapter ends with a selection of exercise statements ranging in difficulty from the direct application of the theory to dealing with some topics that go beyond it. The reader will also find several illuminating examples. We have included a chapter with a list of further reading outlining the different directions in which differential Galois theory and related topics are being developed.

As guidance for teachers interested in using this book for a postgraduate course, we propose three possible courses, depending on the background and interests of their students.

(1) For students with limited or no knowledge of algebraic geometry who wish to understand Galois theory of linear differential equations in all its depth, a two–semester course can be given using the whole book.

(2) For students with good knowledge of algebraic geometry and algebraic groups, a one–semester course can be given based on Part 3 of the book using the first two parts as reference as needed.

(3) For students without a good knowledge of algebraic geometry and eager to learn differential Galois theory more quickly, a one–semester course can be given by developing the topics included in the following sections: 1.1, 3.1, 3.2, 3.3, 4.4 (skipping the references to Lie algebra), 4.6, and Part 3 (except the proof of Proposition 6.3.5, i.e. that the intermediate field of a Picard-Vessiot extension fixed by a normal closed subgroup of the differential Galois group is a Picard-Vessiot extension of the base field). This means introducing the concept of affine variety, defining the algebraic group and its properties considering only affine ones, determining the subgroups of $\mathrm{SL}(2,\mathbb{C})$ assuming as a fact that a connected linear group of dimension less than or equal to 2 is solvable, and developing differential Galois theory (skipping the proof of Proposition 6.3.5).

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Index

Differential Galois theory has seen intense research activity during the last decades in several directions: elaboration of more general theories, computational aspects, model theoretic approaches, applications to classical and quantum mechanics as well as to other mathematical areas such as number theory.

This book intends to introduce the reader to this subject by presenting Picard-Vessiot theory, i.e. Galois theory of linear differential equations, in a self-contained way. The needed prerequisites from algebraic geometry and algebraic groups are contained in the first two parts of the book. The third part includes Picard-Vessiot extensions, the fundamental theorem of Picard-Vessiot theory, solvability by quadratures, Fuchsian equations, monodromy group and Kovacic's algorithm. Over one hundred exercises will help to assimilate the concepts and to introduce the reader to some topics beyond the scope of this book.

This book is suitable for a graduate course in differential Galois theory. The last chapter contains several suggestions for further reading encouraging the reader to enter more deeply into different topics of differential Galois theory or related fields.

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