Riemann Surfaces by Way of Complex Analytic Geometry

Dror Varolin

Graduate Studies in Mathematics

Volume 125

American Mathematical Society

http://dx.doi.org/10.1090/gsm/125

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Graduate Studies
in Mathematics
Volume 125

American Mathematical Society
Providence, Rhode Island

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2010 *Mathematics Subject Classification.* Primary 30F10, 30F15, 30F30, 30F45, 30F99, 30G99, 31A05, 31A99, 32W05.

For additional information and updates on this book, visit
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Library of Congress Cataloging-in-Publication Data

Varolin, Dror, 1970–
Riemann surfaces by way of complex analytic geometry / Dror Varolin.
p. cm. — (Graduate studies in mathematics ; v. 125)
Includes bibliographical references and index.
ISBN 978-0-8218-5369-6 (alk. paper)
1. Riemann surfaces. 2. Functions of complex variables. 3. Geometry, Analytic. I. Title. II. Series.

QA333.V37 2011
515′.93—dc23

2011014621

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10 9 8 7 6 5 4 3 2 1 16 15 14 13 12 11

Dedicated to Erin

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Preface

What

This book presents certain parts of the basic theory of Riemann surfaces through methods of complex analytic geometry, many of which were developed at one time or another in the past 50 years or so.

The first chapter of the book presents a rapid review of the elementary part of basic complex analysis, introducing holomorphic, meromorphic, harmonic, and subharmonic functions and establishing some of the basic local properties of these functions. In the second chapter we define Riemann surfaces and give some examples, both of a concrete and abstract nature. The most abstract example is that any oriented 2-dimensional real manifold has a complex structure. In fact, we state the classical theorem of Korn-Lichtenstein: given any metric on an oriented surface, there is a complex structure that makes that metric locally conformal to the Euclidean metric in any holomorphic chart. The proof of this theorem is given only in Chapter 11.

Riemann surfaces are non-linear spaces on which, roughly speaking, local complex analysis makes sense, and so we can extend the inherently local notions of holomorphic, meromorphic, harmonic, and subharmonic functions to such surfaces. This point of view starts the third chapter, but by the end we prove our first main result, namely the constancy of the degree of a holomorphic map between two compact Riemann surfaces. With the recollection of some basic topology of surfaces, we are then able to establish the Riemann-Hurwitz Formula, which puts constraints on holomorphic maps between compact Riemann surfaces. The chapter ends with a proof of the Harnack Principle. Up to this point, the presentation is mostly classical, in the sense that it does not differ markedly from other presentations of the same material (though perhaps the particular blend of the ideas is unusual).

The novelty of the book begins in the fourth chapter, where we first define and give examples of complex and holomorphic line bundles. There are a number of important, albeit elementary, ideas that are presented in this chapter, and among these the correspondence between divisors and holomorphic line bundles with sections is first established. Another key idea is the correspondence between finite-dimensional vector spaces of sections of a holomorphic line bundle and projective maps. The chapter ends with an elementary proof of the finite-dimensionality of the space of all global holomorphic sections of a holomorphic line bundle over a compact Riemann surface.

The fifth chapter begins by defining complex differential forms. These forms play a central role in the rest of the book, but in this particular chapter we focus on their formal aspects—how to integrate them and how to describe the topology of a Riemann surface with them.

In the sixth chapter we develop the theory of differentiation for sections of complex line bundles (which is known as *defining a connection*). A complex line bundle admits many connections and, unlike the trivial bundle, there is no canonical choice of a connection for a general complex line bundle. The first goal of the chapter is to describe geometric structures that isolate a subset of the possible ways to differentiate sections. If a geometry is symmetric enough, there will be a canonical way to define differentiation of sections. The first instance of such a scenario is the classical Levi-Čivita Theorem for vector fields on a Riemannian manifold. The main fact we establish in Chapter 6, due to Chern, is that for each holomorphic line bundle with Hermitian metric there is a unique connection that is compatible with these structures in the appropriate sense. We also define the curvature of a connection, though we postpone a thorough demonstration of how curvature got its name. We demonstrate Chern's crucial observation that average curvature captures topological information.

We begin the seventh chapter with a discussion of potential theory. We use Perron's Method to solve the Dirichlet Problem and, what is almost the same thing, define a Green's Function on any Riemann surface that admits bounded non-constant subharmonic functions. Our use of the Green's Function is, to our knowledge, somewhat novel; we obtain a Cauchy-Green-type representation formula and use it to solve $\bar{\partial}$ on large (so topologically non-trivial) relatively compact domains with smooth boundary in a Riemann surface. In the second half we use the method of Runge-Behnke-Stein, together with our solution of $\bar{\partial}$, to approximate any local holomorphic function by global ones on an open Riemann surface, provided the complement of domain of the function to be approximated has no relatively compact components.

What was done in Chapter 7 immediately applies to obtain global solutions of the $\bar{\partial}$-equation on any open Riemann surface, and this is the first main result of Chapter 8. We then use this result to establish a number of classical facts about

holomorphic and meromorphic functions on that Riemann surface. The chapter ends with an adaptation of the method of solving $\bar{\partial}$ to the problem of solving $\partial\bar{\partial}$.

If it were obvious that all Riemann surfaces admit holomorphic line bundles with metrics of positive curvature, our next chapter would be Chapter 11. However, the problem of finding a positive line bundle, especially on a compact Riemann surface, is a little bit subtle. We have three ways to find positive line bundles, and the first of these is linked to the intimate relationship between the Poisson Equation and curvature. We therefore establish in Chapter 9 the celebrated Hodge Theorem, in the case of Riemann surfaces. The Hodge Theorem is fundamental in its own right, and the higher-dimensional analog (whose proof is not much different) plays a significant role in algebraic geometry. After establishing the Hodge Theorem, we use it to complete the proofs of two theorems on cohomology of compact Riemann surfaces and to obtain positive line bundles.

Another approach to the existence of positive line bundles is through the Uniformization Theorem of Riemann-Köbe. We therefore establish this theorem in Chapter 10. From a direct observation, one then obtains the positivity of either the tangent or cotangent bundle of any Riemann surface, compact or open, that is not a complex torus. The case of a complex torus is treated in an *ad hoc* manner using Theta functions, which were defined in an example in Chapter 3.

In Chapter 11 we prove Hörmander's Theorem. By this point, we have all the geometric machinery we need, and we set up the Hilbert space method, which is a twisted version of part of the method in the proof of the Hodge Theorem. We then adapt the method to prove the Korn-Lichtenstein Theorem that we stated in Chapter 2.

Chapter 12 concerns the embedding problem for Riemann surfaces. If a Riemann surface is compact without boundary, it cannot be embedded in Euclidean space. (Indeed, if a Riemann surface is embedded in Euclidean space, the coordinate functions of the ambient space restrict to holomorphic functions on the surface, and since the surface is compact without boundary, the real parts (as well as the moduli) of these restricted coordinate functions have interior local maxima. By the Maximum Principle, the restrictions of the coordinate functions are thus locally constant, which contradicts the assumption that the surface is embedded.) The classical remedy for this problem was to puncture a few holes in the surface and embed the punctured surface using meromorphic functions with 'poles in the holes'. There is an interpretation of the resulting embedded surface as lying in projective space, but we take a more direct geometric approach and work directly on projective spaces. It is here that one sees the need for holomorphic line bundles most clearly.

In Chapter 13 we establish the Riemann-Roch Theorem. For an advanced book, our approach is elementary and admittedly less illuminating than a more sheaf-theoretic approach might be. Though the approach looks different from other

proofs, it is simply a combination of two known facts: the usual sheaf-theoretic approach to the proof together with the realization of Serre duality (a result that is stated in the language of sheaf-theory) via residues.

The final chapter of the book states and proves Abel's Theorem, which characterizes linearly trivial divisors, i.e., divisors of meromorphic functions, on a compact Riemann surface. It is elementary to see—as we do on several occasions—that such a divisor must have degree zero and that if the surface in question has positive genus, this necessary condition is not sufficient. Based on computing the periods of global holomorphic forms, Abel constructed a map from the surface into a complex torus whose dimension is the genus of the surface. Abel's map extends to divisors in the obvious way because the torus is an additive group, and the map is a group homomorphism. Abel showed that a necessary and sufficient condition for a divisor to be linearly trivial is that this divisor is mapped to 0 in the torus under Abel's map. The torus is called the Jacobian of the Riemann surface, and Jacobi showed that the set of divisors on the surface maps surjectively onto the Jacobian. We do not prove Jacobi's Theorem, but we do give a sketch of the idea. The chapter, and thus the book, ends with an interpretation of the Abel-Jacobi Theorem as a classification of all holomorphic line bundles on a compact Riemann surface.

Why

The present book arose from the need to bridge what I perceived as a substantial gap between what graduate students at Stony Brook know after they have passed their qualifying exams and higher-dimensional complex analytic geometry in its present state. At present, the generic post-qual student at Stony Brook is relatively well-prepared in algebraic topology and differential geometry but far from so in real and complex analysis or in partial differential equations (though two years after the first draft of this book, the number of students at Stony Brook interested either in algebraic geometry or in partial differential equations has increased significantly; those interested in both still form a very small, nearly empty set).

Courses in complex analysis do not typically emphasize the points most important in the study of Riemann surfaces, focusing instead on functions. For example, the courses focus on direct consequences of Cauchy's Theorem, the Schwarz Lemma, and the Riemann Mapping Theorem.

In this book, the Riemann Mapping Theorem makes way for results based on solving the inhomogeneous Cauchy-Riemann equations (and sometimes Poisson's equation). We present a number of methods for solving these equations, introducing and discussing Green's Functions and Runge-type approximation theorems for this purpose and giving a proof of the Hodge Theorem using basic Hilbert and Sobolev space theory. Perhaps the centerpiece is the Andreotti-Vesentini-Hörmander Theorem on the solution of $\bar{\partial}$ with L^2 estimates. (The theorem has come to be called *Hörmander's Theorem*, though the history is well known and the

presentation we use here is closer in geometric spirit to Andreotti and Vesentini's version of the theorem.) The proof of Hörmander's Theorem in one complex dimension simplifies greatly, most naively because the $\bar{\partial}$-equation does not come with a compatibility condition, and more technically because a certain boundary condition that arises in the functional analytic formulation of the $\bar{\partial}$-problem on Hilbert spaces is a Dirichlet boundary condition, as opposed to its higher-dimensional and more temperamental relative, the $\bar{\partial}$-Neumann boundary condition of Spencer-Kohn-Morrey, which requires the introduction of the notion of pseudo-convexity of the boundary of a domain.

Line bundles play a major role in the book, providing the backdrop and geometric motivation for much of what is done. The 1-dimensional aspect makes the Hermitian geometry rather easy to deal with, and gives the novice a gentle introduction to the higher-dimensional differential geometry of Hermitian line bundles and vector bundles. For example the Kähler condition, which plays such an important role in the higher-dimensional theory, is automatic. Moreover, the meaning of curvature of line bundles, which is beautifully demonstrated in establishing what has come to be known as the Bochner-Kodaira Identity, is greatly simplified in complex dimension 1. (The Bochner-Kodaira Identity is used to estimate from below the smallest eigenvalue of the Laplace-Beltrami operator on sections of a Hermitian holomorphic line bundle.) The identity is obtained through integration by parts. At a certain point one must interchange the order of some exterior differential operators. When a non-flat geometry is present, the commutator of these operators is non-trivial and is the usual definition of curvature. If this commutator, which is a multiplier, is positive, then we obtain a positive lower bound for the smallest eigenvalue of a certain geometric Laplacian, and this lower bound is precisely what is needed to apply the functional analytic method to prove Hörmander's Theorem.

The main application of Hörmander's Theorem is to the existence and plenitude of holomorphic sections of sufficiently positive line bundles. Using these sections, we prove the existence of non-trivial meromorphic functions on Riemann surfaces, non-trivial meromorphic sections of any holomorphic line bundle, and the existence of a projective embedding for any compact Riemann surface—the Kodaira Embedding Theorem. Almost simultaneously, we prove that any open Riemann surface embeds in $\mathbb{C}^3$.

Despite embracing line bundles, I made the choice to avoid both vector bundles and sheaves, two natural extensions of line bundles. This choice shows simultaneously that (i) sheaves and higher-rank vector bundles are not needed in the basic theory of Riemann surfaces and (ii) at times the absence of vector bundles and sheaves makes the presentation cumbersome in certain places. A good example of (i) is the proof of Kodaira embedding without the artifice of sheaf cohomology, but rather through a simple-minded direct construction of certain sections using a beautiful idea first introduced by Bombieri. (Steve Zelditch has coined the perfect

name: *designer sections*.) A reasonable example of (ii) is provided by a number of the results proved in Chapter 8, such as the Mittag-Leffler Theorem. A second example of (ii) is seen in our proof of the Riemann-Roch Theorem without the use of sheaves.

The book ends with two classical results, namely the theorems of Riemann-Roch and Abel. These results constitute an anti-climax of sorts, since they make little use of what was done in the book up to that point. In this regard, we do not add much to what is already in the literature, especially in the case of Abel's Theorem. The inclusion of the two theorems is motivated by seeing them as concluding remarks: the Riemann-Roch Theorem allows one to sharpen the Kodaira Embedding Theorem (or in the language of modern analytic geometry, give an effective embedding result), while Abel's Theorem and its complement, Jacobi's Inversion Theorem, are included because they provide a kind of classification for perhaps the most central group of characters in the book, namely, holomorphic line bundles.

At the urging of many people, I have included exercises for the first 12 chapters of the book. I chose to omit exercises for the last two chapters because, as I have suggested, they do not fall in line with the main pedagogical point of the book, namely, the use of the $\bar{\partial}$- and $\partial\bar{\partial}$-equations.

Of course, there are many glaring omissions that would appear in a standard treatise on Riemann surfaces. We do not discuss Weierstrass points and the finiteness of the automorphism group of a compact Riemann surface of genus at least two. Riemann's Theta Functions are not studied in any great detail. We bring them up only on the torus as a demonstrative tool. As a consequence, we do not discuss Torelli's Theorem. We also omit any serious discussion of basic algebraic geometry of curves (except a few brief remarks mostly scattered around the beginning and end of the book) or of monodromy. There are certainly other omissions, some of which I am not even aware of. Psychologically, the most difficult omission for me was that of a discussion of interpolation and sampling on so-called finite open Riemann surfaces. The theory of interpolation and sampling provides a natural setting (in fact, the only non-trivial natural setting I know in one complex dimension) in which to introduce the twisted $\bar{\partial}$-technique of Ohsawa-Takegoshi. This technique has had incredibly powerful applications in both several complex variables and algebraic geometry, and at the time of the writing of this book there remain many avenues of research to pursue. I chose to omit this topic because, by comparison with the rest of the book, it is disproportionately technical in nature.

Who

The ideal audience for this book consists of students who are interested in analysis and geometry and have had basic first courses in real and complex analysis, differentiable manifolds, and topology. My greatest motivation in writing the book was to help such students in the transition from complex analysis to complex analytic

geometry in higher dimensions, but I hope that the book will find a much wider audience.

In order to get through this book and emerge with a reasonable feeling for the subject, the reader must be at least somewhat prepared in the following sense.

I assume that the reader is well versed in advanced calculus and has seen basic differential topology. For example, the reader should be at ease with the definition of a manifold and the basics of integration of differential forms.

The reader should certainly have taken a first serious course in complex analysis. We state and basically prove all that we need from the early parts of such a course in the first chapter, but the presentation, though fairly complete, is terse and would not be the ideal place to learn the material.

Some minimal amount of real analysis is required in the book. For example, the reader should have seen the most elementary parts of the Riemann and Lebesgue theories of integration. On a couple of occasions we make use of the Hahn-Banach Theorem, the Banach-Alaoglu Theorem, and the Spectral Theorem for Compact, Self-adjoint Operators, but a deep understanding of the proofs of these theorems is not essential.

The topology of a Riemann surface certainly plays a role in much of the book, and in the later chapters the reader encounters a little bit of algebraic topology. Though the notions of homotopy, covering space, and fundamental group are introduced, there is not much detail for the uninitiated, and the reader truly interested in that part of the book should have preparation in those subjects and is moreover probably reading the wrong book on Riemann surfaces.

Very little basic familiarity with linear and, just barely, multilinear algebra is required on the part of the reader. So little is assumed that if the reader is not familiar with some of it but has the mathematical maturity of the aforementioned requirements, there should be no problem in filling the gaps during reading.

How

In my days as a pizza delivery guy for Pizza Pizza in Toronto, I had a colleague named Vlad who used to say in a thick Russian accent: “No money, no funny!” I am grateful to the NSF for its generous financial support.

Much of what is presented in this book is motivated by the work of Jean-Pierre Demailly and Yum-Tong Siu, and I am grateful to both of them for all that they have taught me, both in their writings and in person. John D’Angelo and Jeff McNeal were very encouraging in the early parts of the project and gave me the inspiration I needed to start the project. Andy Raich and Colleen Robles read a preliminary version of these notes at Texas A&M, and Colleen communicated corrections and suggestions that were extremely useful. I am grateful to both of them. Steve Zelditch has used the notes for part of his course on Riemann surfaces and

has sent back comments for which I owe him a debt of gratitude, as does the future reader of this book. The final blow was delivered at the University of Cincinnati, where I was fortunate enough to be the Taft Fellow for the spring quarter of 2010, for which I am grateful to the Charles Phelps Taft Foundation and all those involved in administering various parts of my fellowship. During the tenure of my Taft Fellowship I gave a mini-course, covering some of the material in the text, and most importantly I came up with exercises for the text. I am grateful to David Herron and David Minda for arranging the lectures and my visit. They attended my mini-course, as did Anders and Jana Bjorn, Robbie Buckingham, Andy Lorent, Diego Mejia, Mihaela Poplicher, and Nages Shanmugalingam, and I am grateful to all of them for their patience with my lectures and their generous hospitality.

There were a number of anonymous referees who communicated a number of valuable suggestions and corrections. I would like to thank them all for their service and help and apologize to them if I did not agree with all of their suggestions, perhaps wrongly.

The book would certainly not exist were it not for the efforts of Ina Mette, to whom I warmly express my heartfelt gratitude. Two other people at the AMS, namely Marcia Almeida and Arlene O'Sean, were instrumental in helping me to get the book finished, and I thank them for their help and their kind dealings.

Most of all, I am indebted to Mohan Ramachandran. Our frequent conversations, beyond giving me great pleasure, led to the inclusion of many important topics and to the correction of many errors, and Mohan's passion for the old literature taught me an enormous amount about the history and development of the subject.

Where and when

Dror Varolin
Brooklyn, NY
2010

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After three introductory chapters, the book embarks on its central, and certainly most novel, goal of studying Hermitian holomorphic line bundles and their sections. Among other things, finite-dimensionality of spaces of sections of holomorphic line bundles of compact Riemann surfaces and the triviality of holomorphic line bundles over Riemann surfaces are proved, with various applications. Perhaps the main result of the book is Hörmander's Theorem on the square-integrable solution of the Cauchy-Riemann equations. The crowning application is the proof of the Kodaira and Narasimhan Embedding Theorems for compact and open Riemann surfaces.

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This book is the first to give a textbook exposition of Riemann surface theory from the viewpoint of positive Hermitian line bundles and Hörmander $\bar{\partial}$ estimates. It is more analytical and PDE oriented than prior texts in the field, and is an excellent introduction to the methods used currently in complex geometry, as exemplified in J. P. Demailly's online but otherwise unpublished book "Complex analytic and differential geometry." I used it for a one quarter course on Riemann surfaces and found it to be clearly written and self-contained. It not only fills a significant gap in the large textbook literature on Riemann surfaces but is also rather indispensible for those who would like to teach the subject from a differential geometric and PDE viewpoint.

—***Steven Zelditch***

ISBN 978-0-8218-5369-6

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