Gröbner Bases in Commutative Algebra

Viviana Ene
Jürgen Herzog

Graduate Studies in Mathematics
Volume 130

American Mathematical Society

http://dx.doi.org/10.1090/gsm/130

Gröbner Bases in Commutative Algebra

Viviana Ene
Jürgen Herzog

Graduate Studies in Mathematics
Volume 130

American Mathematical Society
Providence, Rhode Island

2010 *Mathematics Subject Classification.* Primary 13-01, 13A15, 13D02, 13H10, 13P10.

For additional information and updates on this book, visit
www.ams.org/bookpages/gsm-130

Library of Congress Cataloging-in-Publication Data

Ene, Viviana, 1960–
Gröbner bases in commutative algebra / Viviana Ene, Jürgen Herzog.
p. cm. – (Graduate studies in mathematics ; v. 130)
Includes bibliographical references and index.
ISBN 978-0-8218-7287-1 (alk. paper)
1. Gröbner bases. 2. Commutative algebra. I. Herzog, Jürgen, 1947– II. Title.

QA251.3.E54 2012
512′.44–dc23 2011032432

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
Visit the AMS home page at `http://www.ams.org/`

10 9 8 7 6 5 4 3 2 1 17 16 15 14 13 12

To our parents Maria and Ion, Margarete and Walter

Contents

Preface

Gröbner basis theory has become a fundamental field in algebra which provides a wide range of theoretical and computational methods in many areas of mathematics and other sciences. Bruno Buchberger defined the notion of Gröbner basis in 1965 [**Bu65**]. An intensive research in this theory, related algorithms and applications developed, and many books on this topic have appeared since then. Among them, the books of Adams and Loustaunau [**AL94**], Becker, Kredel, Weispfenning [**BKW93**], Cox, Little, O'Shea [**CLO05**], [**CLO07**], and Eisenbud [**E95**] give a fine introduction to Gröbner basis theory and its applications. Many computer algebra systems like CoCoA, Macaulay2, Magma, Maple, Mathematica, or Singular have implemented various versions of Buchberger's algorithm.

This book aims to provide a concise but rather comprehensive introduction to the theory of Gröbner bases and to introduce the reader to different current trends in theoretical applications of this theory. The complexity level of the presentation increases gradually. The first three chapters and part of Chapter 4 are self-contained and lead to a quick insight into the basics of Gröbner basis theory. They require only a very basic knowledge of algebra. Therefore, this first part of the book would also be appropriate for those readers who are only familiar with elementary algebraic concepts. The second part of Chapter 4 and the last two chapters discuss more advanced topics related to the theory together with applications of it and require a reasonable knowledge in commutative and homological algebra.

Our purpose in writing this book was to provide young researchers and graduate students in commutative algebra and algebraic geometry with methods and techniques related to the Gröbner basis theory. Although it was not our goal to illustrate the algorithmic and computational attributes of the Gröbner basis theory, the usage of the computer in testing examples is indispensable. Users of computer algebra systems may consult specialized monographs like [**GP02**], [**Mac01**] or [**KR00**], [**KR05**].

We give now a brief summary of the book's content. Chapter 1 presents polynomial rings in finitely many indeterminates over a field together with their basic properties and studies ideals in this class of rings. The last two sections are devoted to monomial ideals and standard operations on them. Chapter 2 provides a short but comprehensive exposition of the Gröbner basis notion and Buchberger's criterion and algorithm. In Chapter 3 we discuss first applications based on the Elimination Theorem. Chapter 4 is devoted to the extension of the Gröbner basis theory to submodules of free modules over polynomial rings. The chapter begins with a quick introduction to module theory. The more general concepts discussed here lead to a proof, due to Schreyer, for the celebrated Hilbert's Syzygy Theorem. Chapter 5 opens the series of applications of Gröbner basis theory in commutative algebra and combinatorics. In this chapter we discuss semigroup rings and toric ideals which are intensively studied nowadays from different points of view. Chapter 6 intends to introduce the reader to more advanced topics and to subjects of recent research. The topics treated in this section are not presented in the largest possible generality. Instead, one of the main goals of this last chapter of this monograph is to inspire and to enable the reader, who is interested in further developments and other aspects of the theory, to read more advanced monographs and articles on these subjects, for example, the monograph of Sturmfels [**St95**] which influenced the writing of this chapter substantially, the book [**MS05**] of Miller and Sturmfels, the influential monograph [**S96**] of Stanley and the book of Hibi [**Hi92**]. A compact and detailed presentation of determinantal ideals can be found in the article of Bruns and Conca [**BC03**]. A reader who is interested in further results on monomial ideals should consult the book [**V01**] of Villarreal and the book [**HH10**] of Herzog and Hibi. Further references to research articles are given in the text of Chapter 6. In the first section of the chapter Gröbner bases are used to study Koszul algebras. In particular, it is shown that algebras whose defining ideal has a quadratic Gröbner basis are Koszul. Large classes of algebras whose defining ideal has a quadratic Gröbner basis are provided by algebras generated by sortable sets. This is the topic of the next section. The theory of sortable sets is then applied in the following sections to study generalized Hibi rings and Rees algebras. Next we outline the approach to the theory of determinantal ideals via Gröbner basis with the conclusion that these ideals are all Cohen–Macaulay. Then we give a short introduction to Sagbi bases and apply the theory to show that the coordinate ring of the Grassmannian of m-dimensional vector K-subspaces of K^n is a Gorenstein ring and compute its dimension. The last two sections of the chapter deal with binomial edge ideals and some aspects of algebraic statistics.

The book contains over one hundred problems with a moderate level of difficulty which illuminate the theory and help the reader to fully understand the results discussed in the text. Some of them complete the proofs. Other problems are more complex and encourage the reader to re-examine the simple but essential ideas, to establish connections, and to become interested in further reading.

We wish to express our thanks to Marius Vlădoiu for the careful reading of earlier drafts of this book.

Viviana Ene and Jürgen Herzog

Bibliography

[AL94] W. W. Adams, P. Loustaunau, *An Introduction to Gröbner bases*, Amer. Math. Soc., Providence, RI, 1994.

[BKW93] T. Becker, H. Kredel, V. Weispfenning, *Gröbner bases: a computational approach to commutative algebra*, Springer, 1993.

[BC03] W. Bruns, A. Conca, *Gröbner bases and determinantal ideals*, In: Commutative Algebra, Singularities ansd Computer Algebra, J. Herzog and V. Vuletescu, Eds., NATO Science Series **115**, (2003), 9–66.

[BG09] W. Bruns, J. Gubeladze, *Polytopes, Rings, and K-theory*, Springer Monographs in Mathematics, Springer, 2009.

[BH98] W. Bruns, J. Herzog, *Cohen-Macaulay rings*, Revised Ed., Cambridge University Press, 1998.

[BV88] W. Bruns, U. Vetter, *Determinantal rings*, Lect. Notes Math. **1327**, Springer, 1988.

[Bu65] B. Buchberger, *Ein Algorithmus zum Auffinden der Basiselemente des Restklassenrings*, PhD thesis, Innsbruck, 1965.

[CLO07] D. Cox, J. Little, D. O'Shea, *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, Third edition, Springer, 2005.

[CLO05] D. Cox, J. Little, D. O'Shea, *Using Algebraic Geometry*, Second edition, Springer, 2005.

[DES98] P. Diaconis, D. Eisenbud, B. Sturmfels, *Lattice walks and primary decomposition*, Mathematical Essays in Honor of Gian-Carlo Rota, Birkhäuser, Boston, 1998, pp. 173–193.

[E95] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics **150**, Springer, 1995.

[EHV92] D. Eisenbud, C. Huneke, W. Vasconcelos, *Direct methods for primary decomposition*, Invent. Math. **110** (1992), 207–235.

[ES84] D. Eisenbud, B. Sturmfels, *Binomial ideals*, Duke Math. J. **84** (1996), 1–45.

[EHM10] V. Ene, J. Herzog, F. Mohammadi, *Monomial ideals and toric rings of Hibi type arising from a finite poset*, Eur. J. Comb. **32** (2011), 404-421.

[FH11] G. Fløystad, J. Herzog, *Gröbner bases of syzygies and Stanley depth*, J. Algebra **328** (2011), 178–189.

[GTZ88] P. Gianni, B. Trager, G. Zacharias, *Gröbner bases and primary decomposition of polynomials ideals*, J. Symb. Comput. **6** (1988), 149–167.

[G87] G. Glonek, *Some aspects of log linear models*, Thesis, School of Math. Sci., Flinders Univ. of S. Australia, 1987.

[GP02] G.-M. Greuel, G. Pfister, *A Singular Introduction to Commutative Algebra*, Springer, 2002.

[HH05] J. Herzog, T. Hibi, *Distributive lattices, bipartite graphs and Alexander duality,* J. Algebraic Combin. **22** (2005), 289–302.

[HH10] J. Herzog, T. Hibi, *Monomial ideals*, Graduate Texts in Mathematics **260**, Springer, 2010.

[HH11] J. Herzog, T. Hibi, *Ideals generated by* 2*-adjacent minors*, preprint 2011.

[HHHKR10] J. Herzog, T. Hibi, F. Hreinsdotir, T. Kahle, J. Rauh, *Binomial edge ideals and conditional independence statements*, Adv. Appl. Math. **45** (2010), 317–333.

[HHR00] J. Herzog, T. Hibi, G. Restuccia, *Strongly Koszul algebras*, Math. Scand. **86** (2000), no. 2, 161–178.

[HHV05] J. Herzog, T. Hibi, M. Vladoiu, *Ideals of fiber type and polymatroids*, Osaka J. Math. **42** (2005), 807-829.

[HT96] J. Herzog, N. V. Trung, *Gröbner bases and multiplicity of determinantal and Pfaffian ideals*, Adv. in Math. **96** (1992), 1–37.

[Hi87] T. Hibi, *Distributive lattices, affine semigroup rings and algebras with straightening laws*, Commutative Algebra and Combinatorics (M. Nagata and H. Matsumura, Eds.) Adv. Stud. Pure Math. **11**, North-Holland, Amsterdam, 1987, 93–109.

[Hi92] T. Hibi, *Algebraic Combinatorics on Convex Polytopes*, Carslaw, Glebe, N.S.W., Australia, 1992.

[Ho72] M. Hochster, *Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes*, Ann. of Math. **96** (1972), 228–235.

[Ho77] M. Hochster, *Cohen-Macaulay rings, combinatorics, and simplicial complexes*, Proceedings of the Second Oklahoma Ring Theory Conference (March 1976), Marcel-Dekker, New York, 1977, 171-223.

[HS00] S. Hoşten, J. Shapiro, *Primary decomposition of lattice basis ideal*, J.Symbolic Computation **29** (2000), 625–639.

[HS04] S. Hoşten, S. Sullivant, *Ideals of adjacent minors*, J. Algebra **277** (2004), 615–642.

[KM89] D. Kapur, K. Madlener, *A completion procedure for computing a canonical basis for a k-subalgebra*, Computer and Mathematics (Cambridge MA 1989), Springer 1989, 1–11.

[KH06] M. Kokubo, T. Hibi, *Weakly polymatroidal ideals*, Alg. Colloq. **13** (2006), 711-720.

[KR00] M. Kreuzer, L. Robbiano, *Computational Commutative Algebra*, Vol. 1, Springer, 2000.

[KR05] M. Kreuzer, L. Robbiano, *Computational Commutative Algebra*, Vol. 2, Springer, 2005.

[HO99] H. Ohsugi, T. Hibi, *Toric ideals generated by quadratic binomials*, J. Algebra **218** (1999), 509–527.

[Mac01] *Computations in algebraic geometry with Macaulay 2*, D. Eisenbud, D. R. Grayson, M. E. Stillman, B. Sturmfels, Eds., Algorithms and Computations in Mathematics **8**, Springer, 2001.

[M86] H. Matsumura, *Commutative ring theory*, Cambridge University Press, 1986.

[Mi10] E. Miller, *Theory and applications of Lattice Point Methods for Binomial Ideals*, in Combinatorial aspects of Commutative Algebra and Algebraic Geometry, Abel Symposium, 2009.

[MS05] E. Miller, B. Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics **227**, Springer, 2005.

[O10] M. Ohtami, *Graphs and ideals generated by some* 2*-minors*, Commun Algebra **39**(3) (2011), 905–917.

[R76] G. A. Reisner, *Cohen-Macaulay quotients of polynomial rings*, Adv. in Math. **21** (1976), 30–49.

[RS90] L. Robbiano, M. Sweedler, *Subalgebra bases*, Commutative Algebra (W. Bruns, A. Simis, Eds.) Lect. Notes Math. **1430**, Springer, 1990, 61–87.

[Sc80] F.-O. Schreyer, *Die Berechnung von Syzygien mit dem verallgemeinerten Weierstrass'schen Divisionssatz*, Diploma Thesis, University of Hamburg, Germany.

[S75] R. Stanley, *The upper bound conjecture and Cohen-Macaulay rings*, Stud. Appl. Math. **54** (1975), 135–142.

[S96] R. Stanley, *Combinatorics and commutative algebra*, Second edition, Progress in Mathematics, **41**, Birkhuser Boston, Inc., Boston, MA, 1996.

[St90] B. Sturmfels, *Gröbner bases and Stanley decompositions of determinantal rings*, Math. Z. **205** (1990), 137–144.

[St95] B. Sturmfels, *Gröbner Bases and Convex Polytopes*, Amer. Math. Soc., Providence, RI, 1995.

[V90] R. Villarreal, *Cohen-Macaulay graphs*, Manuscripta Math. **66** (1990), 277–293.

[V01] R.H. Villarreal, *Monomial algebras*, Marcel Dekker, 2001.

Index

This book provides a concise yet comprehensive and self-contained introduction to Gröbner basis theory and its applications to various current research topics in commutative algebra. It especially aims to help young researchers become acquainted with fundamental tools and techniques related to Gröbner bases which are used in commutative algebra and to arouse their interest in exploring further topics such as toric rings, Koszul and Rees algebras, determinantal ideal theory, binomial edge ideals, and their applications to statistics.

The book can be used for graduate courses and self-study. More than 100 problems will help the readers to better understand the main theoretical results and will inspire them to further investigate the topics studied in this book.

ISBN 978-0-8218-7287-1

9 780821 872871

GSM/130

For additional information and updates on this book, visit

www.ams.org/bookpages/gsm-130