

# Topics in Random Matrix Theory

**Terence Tao**

**Graduate Studies  
in Mathematics**

**Volume 132**



**American Mathematical Society**

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To Garth Gaudry, who set me on the road;  
To my family, for their constant support;  
And to the readers of my blog, for their feedback and contributions.



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# Preface

In the winter of 2010, I taught a topics graduate course on random matrix theory, the lecture notes of which then formed the basis for this text. This course was inspired by recent developments in the subject, particularly with regard to the rigorous demonstration of universal laws for eigenvalue spacing distributions of Wigner matrices (see the recent survey [Gu2009b]). This course does not directly discuss these laws, but instead focuses on more foundational topics in random matrix theory upon which the most recent work has been based. For instance, the first part of the course is devoted to basic probabilistic tools such as concentration of measure and the central limit theorem, which are then used to establish basic results in random matrix theory, such as the Wigner semicircle law on the bulk distribution of eigenvalues of a Wigner random matrix, or the circular law on the distribution of eigenvalues of an iid matrix. Other fundamental methods, such as free probability, the theory of determinantal processes, and the method of resolvents, are also covered in the course.

This text begins in Chapter 1 with a review of the aspects of probability theory and linear algebra needed for the topics of discussion, but assumes some existing familiarity with both topics, as well as a first-year graduate-level understanding of measure theory (as covered for instance in my books [Ta2011, Ta2010]). If this text is used to give a graduate course, then Chapter 1 can largely be assigned as reading material (or reviewed as necessary), with the lectures then beginning with Section 2.1.

The core of the book is Chapter 2. While the focus of this chapter is ostensibly on random matrices, the first two sections of this chapter focus more on random *scalar* variables, in particular, discussing extensively the concentration of measure phenomenon and the central limit theorem in this

setting. These facts will be used repeatedly when we then turn our attention to random matrices, and also many of the proof techniques used in the scalar setting (such as the moment method) can be adapted to the matrix context. Several of the key results in this chapter are developed through the exercises, and the book is designed for a student who is willing to work through these exercises as an integral part of understanding the topics covered here.

The material in Chapter 3 is related to the main topics of this text, but is optional reading (although the material on Dyson Brownian motion from Section 3.1 is referenced several times in the main text).

This text is *not* intended as a comprehensive introduction to random matrix theory, which is by now a vast subject. For instance, only a small amount of attention is given to the important topic of invariant matrix ensembles, and we do not discuss connections between random matrix theory and number theory, or to physics. For these topics we refer the reader to other texts such as [AnGuZi2010], [DeGi2007], [De1999], [Fo2010], [Me2004]. We hope, however, that this text can serve as a foundation for the reader to then tackle these more advanced texts.

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I am greatly indebted to my students of the course on which this text was based, as well as many further commenters on my blog, including Ahmet Arivan, Joshua Batson, Florent Benaych-Georges, Sivaraman Balakrishnan, Alex Bloemendal, Kaihua Cai, Andres Caicedo, Emmanuel Candés, Jérôme Chauvet, Brian Davies, Ben Golub, Stephen Heilman, John Jiang, Li Jing, Rowan Killip, Sungjin Kim, Allen Knutson, Greg Kuperberg, Choongbum Lee, George Lowther, Rafe Mazzeo, Mark Meckes, William Meyerson, Samuel Monnier, Andreas Naive, Srivatsan Narayanan, Giovanni Peccati, Leonid Petrov, Anand Rajagopalan, Brian Simanek, James Smith, Mads Sørensen, David Speyer, Ambuj Tewari, Luca Trevisan, Qiaochu Yuan, and several anonymous contributors, for comments and corrections. These comments, as well as the original lecture notes for this course, can be viewed online at:

[terrytao.wordpress.com/category/teaching/254a-random-matrices](http://terrytao.wordpress.com/category/teaching/254a-random-matrices)

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Last, but not least, I am indebted to my co-authors Emmanuel Candés and Van Vu, for introducing me to the fascinating world of random matrix theory, and to the anonymous referees of this text for valuable feedback and suggestions.



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