

# Hyperbolic Partial Differential Equations and Geometric Optics

**Jeffrey Rauch**

**Graduate Studies  
in Mathematics**

**Volume 133**



**American Mathematical Society**

# Hyperbolic Partial Differential Equations and Geometric Optics



# Hyperbolic Partial Differential Equations and Geometric Optics

Jeffrey Rauch

Graduate Studies  
in Mathematics

Volume 133



American Mathematical Society  
Providence, Rhode Island

## EDITORIAL COMMITTEE

David Cox (Chair)  
Rafe Mazzeo  
Martin Scharlemann  
Gigliola Staffilani

2010 *Mathematics Subject Classification*. Primary 35A18, 35A21, 35A27, 35A30, 35Q31, 35Q60, 78A05, 78A60, 78M35, 93B07.

---

For additional information and updates on this book, visit  
[www.ams.org/bookpages/gsm-133](http://www.ams.org/bookpages/gsm-133)

---

### Library of Congress Cataloging-in-Publication Data

Rauch, Jeffrey.

Hyperbolic partial differential equations and geometric optics / Jeffrey Rauch.

p. cm. — (Graduate studies in mathematics ; v. 133)

Includes bibliographical references and index.

ISBN 978-0-8218-7291-8 (alk. paper)

1. Singularities (Mathematics). 2. Microlocal analysis. 3. Geometrical optics—Mathematics.  
4. Differential equations, Hyperbolic. I. Title.

QC20.7.S54R38 2012  
535'.32—dc23

2011046666

---

**Copying and reprinting.** Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy a chapter for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication is permitted only under license from the American Mathematical Society. Requests for such permission should be addressed to the Acquisitions Department, American Mathematical Society, 201 Charles Street, Providence, Rhode Island 02904-2294 USA. Requests can also be made by e-mail to [reprint-permission@ams.org](mailto:reprint-permission@ams.org).

© 2012 by Jeffrey Rauch. All rights reserved.

The American Mathematical Society retains all rights  
except those granted to the United States Government.

Printed in the United States of America.

⊗ The paper used in this book is acid-free and falls within the guidelines  
established to ensure permanence and durability.

Visit the AMS home page at <http://www.ams.org/>

10 9 8 7 6 5 4 3 2 1 17 16 15 14 13 12

To Geraldine.



---

# Contents

Preface	xi
§P.1. How this book came to be, and its peculiarities	xi
§P.2. A bird's eye view of hyperbolic equations	xiv
Chapter 1. Simple Examples of Propagation	1
§1.1. The method of characteristics	2
§1.2. Examples of propagation of singularities using progressing waves	12
§1.3. Group velocity and the method of nonstationary phase	16
§1.4. Fourier synthesis and rectilinear propagation	20
§1.5. A cautionary example in geometric optics	27
§1.6. The law of reflection	28
1.6.1. The method of images	30
1.6.2. The plane wave derivation	33
1.6.3. Reflected high frequency wave packets	34
§1.7. Snell's law of refraction	36
Chapter 2. The Linear Cauchy Problem	43
§2.1. Energy estimates for symmetric hyperbolic systems	44
§2.2. Existence theorems for symmetric hyperbolic systems	52
§2.3. Finite speed of propagation	58
2.3.1. The method of characteristics.	58
2.3.2. Speed estimates uniform in space	59
2.3.3. Time-like and propagation cones	64
§2.4. Plane waves, group velocity, and phase velocities	71



---

§2.5. Precise speed estimate	79
§2.6. Local Cauchy problems	83
Appendix 2.I. Constant coefficient hyperbolic systems	84
Appendix 2.II. Functional analytic proof of existence	89
Chapter 3. Dispersive Behavior	91
§3.1. Orientation	91
§3.2. Spectral decomposition of solutions	93
§3.3. Large time asymptotics	96
§3.4. Maximally dispersive systems	104
3.4.1. The $L^1 \rightarrow L^\infty$ decay estimate	104
3.4.2. Fixed time dispersive Sobolev estimates	107
3.4.3. Strichartz estimates	111
Appendix 3.I. Perturbation theory for semisimple eigenvalues	117
Appendix 3.II. The stationary phase inequality	120
Chapter 4. Linear Elliptic Geometric Optics	123
§4.1. Euler's method and elliptic geometric optics with constant coefficients	123
§4.2. Iterative improvement for variable coefficients and nonlinear phases	125
§4.3. Formal asymptotics approach	127
§4.4. Perturbation approach	131
§4.5. Elliptic regularity	132
§4.6. The Microlocal Elliptic Regularity Theorem	136
Chapter 5. Linear Hyperbolic Geometric Optics	141
§5.1. Introduction	141
§5.2. Second order scalar constant coefficient principal part	143
5.2.1. Hyperbolic problems	143
5.2.2. The quasiclassical limit of quantum mechanics	149
§5.3. Symmetric hyperbolic systems	151
§5.4. Rays and transport	161
5.4.1. The smooth variety hypothesis	161
5.4.2. Transport for $L = L_1(\partial)$	166
5.4.3. Energy transport with variable coefficients	173
§5.5. The Lax parametrix and propagation of singularities	177
5.5.1. The Lax parametrix	177
5.5.2. Oscillatory integrals and Fourier integral operators	180

---

5.5.3.	Small time propagation of singularities	188
5.5.4.	Global propagation of singularities	192
§5.6.	An application to stabilization	195
Appendix 5.I.	Hamilton–Jacobi theory for the eikonal equation	206
5.I.1.	Introduction	206
5.I.2.	Determining the germ of $\phi$ at the initial manifold	207
5.I.3.	Propagation laws for $\phi, d\phi$	209
5.I.4.	The symplectic approach	212
Chapter 6.	The Nonlinear Cauchy Problem	215
§6.1.	Introduction	215
§6.2.	Schauder’s lemma and Sobolev embedding	216
§6.3.	Basic existence theorem	222
§6.4.	Moser’s inequality and the nature of the breakdown	224
§6.5.	Perturbation theory and smooth dependence	227
§6.6.	The Cauchy problem for quasilinear symmetric hyperbolic systems	230
6.6.1.	Existence of solutions	231
6.6.2.	Examples of breakdown	237
6.6.3.	Dependence on initial data	239
§6.7.	Global small solutions for maximally dispersive nonlinear systems	242
§6.8.	The subcritical nonlinear Klein–Gordon equation in the energy space	246
6.8.1.	Introductory remarks	246
6.8.2.	The ordinary differential equation and non-lipshitzean $F$	248
6.8.3.	Subcritical nonlinearities	250
Chapter 7.	One Phase Nonlinear Geometric Optics	259
§7.1.	Amplitudes and harmonics	259
§7.2.	Elementary examples of generation of harmonics	262
§7.3.	Formulating the ansatz	263
§7.4.	Equations for the profiles	265
§7.5.	Solving the profile equations	270
Chapter 8.	Stability for One Phase Nonlinear Geometric Optics	277
§8.1.	The $H_\epsilon^s(\mathbb{R}^d)$ norms	278
§8.2.	$H_\epsilon^s$ estimates for linear symmetric hyperbolic systems	281
§8.3.	Justification of the asymptotic expansion	282

---

§8.4. Rays and nonlinear transport	285
Chapter 9. Resonant Interaction and Quasilinear Systems	291
§9.1. Introduction to resonance	291
§9.2. The three wave interaction partial differential equation	294
§9.3. The three wave interaction ordinary differential equation	298
§9.4. Formal asymptotic solutions for resonant quasilinear geometric optics	302
§9.5. Existence for quasiperiodic principal profiles	307
§9.6. Small divisors and correctors	310
§9.7. Stability and accuracy of the approximate solutions	313
§9.8. Semilinear resonant nonlinear geometric optics	314
Chapter 10. Examples of Resonance in One Dimensional Space	317
§10.1. Resonance relations	317
§10.2. Semilinear examples	321
§10.3. Quasilinear examples	327
Chapter 11. Dense Oscillations for the Compressible Euler Equations	333
§11.1. The 2 – $d$ isentropic Euler equations	333
§11.2. Homogeneous oscillations and many wave interaction systems	336
§11.3. Linear oscillations for the Euler equations	338
§11.4. Resonance relations	341
§11.5. Interaction coefficients for Euler's equations	343
§11.6. Dense oscillations for the Euler equations	346
11.6.1. The algebraic/geometric part	346
11.6.2. Construction of the profiles	347
Bibliography	351
Index	359

---

# Preface

## P.1. How this book came to be, and its peculiarities

This book presents an introduction to hyperbolic partial differential equations. A major subtheme is linear and nonlinear geometric optics. The two central results of linear microlocal analysis are derived from geometric optics. The treatment of nonlinear geometric optics gives an introduction to methods developed within the last twenty years, including a rethinking of the linear case.

Much of the material has grown out of courses that I have taught. The crucial step was a series of lectures on nonlinear geometric optics at the Institute for Advanced Study/Park City Mathematics Institute in July 1995. The Park City notes were prepared with the assistance of Markus Keel and appear in [Rauch, 1998]. They presented a straight line path to some theorems in nonlinear geometric optics. Graduate courses at the University of Michigan in 1993 and 2008 were important. Much of the material was refined in invited minicourses:

- École Normale Supérieure de Cachan, 1997;
- Nordic Conference on Conservation Laws at the Mittag-Leffler Institute and KTH in Stockholm, December 1997 (Chapters 9–11);
- Centro di Ricerca Matematica Ennio De Giorgi, Pisa, February 2004;
- Université de Provence, Marseille, March 2004 (§3.4, 5.4, Appendix 2.I);
- Università di Pisa, February–May 2005, March–April 2006 (Chapter 3, §6.7, 6.8), March–April 2007 (Chapters 9–11);
- Université de Paris Nord, February 2006–2010 (§1.4–1.7).

The auditors included many at the beginnings of their careers, and I would like to thank in particular R. Carles, E. Dumas, J. Bronski, J. Colliander, M. Keel, L. Miller, K. McLaughlin, R. McLaughlin, H. Zag, G. Crippa, A. Figalli, and N. Visciglia for many interesting questions and comments.

The book is aimed at the level of graduate students who have studied one hard course in partial differential equations. Following the lead of the book of Guillemin and Pollack (1974), there are exercises scattered throughout the text. The reader is encouraged to read with paper and pencil in hand, filling in and verifying as they go. There is a big difference between passive reading and active acquisition. In a classroom setting, correcting students' exercises offers the opportunity to teach the writing of mathematics.

To shorten the treatment and to avoid repetition with a solid partial differential equations course, basic material such as the fundamental solution of the wave equation in low dimensions is not presented. Naturally, I like the treatment of that material in my book *Partial Differential Equations* [Rauch, 1991].

The choice of subject matter is guided by several principles. By restricting to symmetric hyperbolic systems, the basic energy estimates come from integration by parts. The majority of examples from applications fall under this umbrella.

The treatment of constant coefficient problems does not follow the usual path of describing classes of operators for which the Cauchy problem is weakly well posed. Such results are described in Appendix 2.I along with the Kreiss matrix theorem. Rather, the Fourier transform is used to analyse the dispersive properties of constant coefficient symmetric hyperbolic equations including Brenner's theorem and Strichartz estimates.

Pseudodifferential operators are neither presented nor used. This is not because they are in any sense vile, but to get to the core without too many pauses to develop machinery. There are several good sources on pseudodifferential operators and the reader is encouraged to consult them to get alternate viewpoints on some of the material. In a sense, the expansions of geometric optics are a natural replacement for that machinery. Lax's parametrix and Hörmander's microlocal propagation of singularities theorem require the analysis of oscillatory integrals as in the theory of Fourier integral operators. The results require only the method of nonstationary phase and are included.

The topic of caustics and caustic crossing is not treated. The sharp linear results use more microlocal machinery and the nonlinear analogues are topics of current research. The same is true for supercritical nonlinear geometric optics which is not discussed. The subjects of dispersive and

diffractive nonlinear geometric optics in contrast have reached a mature state. Readers of this book should be in a position to readily attack the papers describing that material.

The methods of geometric optics are presented as a way to understand the qualitative behavior of partial differential equations. Many examples proper to the theory of partial differential equations are discussed in the text, notably the basic results of microlocal analysis. In addition two long examples, stabilization of waves in §5.6 and dense oscillations for inviscid compressible fluid flow in Chapter 11 are presented. There are many important examples in science and technology. Readers are encouraged to study some of them by consulting the literature. In the scientific literature there will not be theorems. The results of this book turn many seemingly ad hoc approximate methods into rigorous asymptotic analyses.

Only a few of the many important hyperbolic systems arising in applications are discussed. I recommend the books [Courant, 1962], [Benzoni-Gavage and Serre, 2007], and [Métivier, 2009]. The asymptotic expansions of geometric optics explain the physical theory, also called geometric optics, describing the rectilinear propagation, reflection, and refraction of light rays. A brief discussion of the latter ideas is presented in the introductory chapter that groups together elementary examples that could be, but are usually not, part of a partial differential equations course. The WKB expansions of geometric optics also play a crucial role in understanding the connection of classical and quantum mechanics. That example, though not hyperbolic, is presented in §5.2.2.

The theory of hyperbolic mixed initial boundary value problems, a subject with many interesting applications and many difficult challenges, is not discussed. Nor is the geometric optics approach to shocks.

I have omitted several areas where there are already good sources; for example, the books [Smoller, 1983], [Serre, 1999], [Serre, 2000], [Dafermos, 2010], [Majda, 1984], [Bressan, 2000] on conservation laws, and the books [Hörmander, 1997] and [Taylor, 1991] on the use of pseudodifferential techniques in nonlinear problems. Other books on hyperbolic partial differential equations include [Hadamard, 1953], [Leray, 1953], [Mizohata, 1965], and [Benzoni-Gavage and Serre, 2007]. Lax's 1963 Stanford notes occupy a special place for me. I took a course from him in the late 1960s that corresponded to the enlarged version [Lax, 2006]. When I approached him to ask if he'd be my thesis director he asked what interested me. I indicated two subjects from the course, mixed initial boundary value problems and the section on waves and rays. The first became the topic of my thesis, and the second is the subject of this book and at the core of much of my research. I

owe a great intellectual debt to the lecture notes, and to all that Peter Lax has taught me through the years.

The book introduces a large and rich subject and I hope that readers are sufficiently attracted to probe further.

## P.2. A bird's eye view of hyperbolic equations

The central theme of this book is hyperbolic partial differential equations. These equations are important for a variety of reasons that we sketch here and that recur in many different guises throughout the book.

The first encounter with hyperbolicity is usually in considering scalar real linear second order partial differential operators in two variables with coefficients that may depend on  $x$ ,

$$a u_{x_1 x_1} + b u_{x_1 x_2} + c u_{x_2 x_2} + \text{lower order terms}.$$

Associate the quadratic form  $\xi \mapsto a \xi_1^2 + b \xi_1 \xi_2 + c \xi_2^2$ . The differential operator is *elliptic* when the form is positive or negative definite. The differential operator is *strictly hyperbolic* when the form is indefinite and nondegenerate.<sup>1</sup>

In the elliptic case one has strong local regularity theorems and solvability of the Dirichlet problem on small discs. In the hyperbolic cases, the initial value problem is locally well set with data given at noncharacteristic curves and there is finite speed of propagation. Singularities or oscillations in Cauchy data propagate along characteristic curves.

The defining properties of hyperbolic problems include well posed Cauchy problems, finite speed of propagation, and the existence of wave like structures with infinitely varied form. To see the latter, consider in  $\mathbb{R}_{t,x}^2$  initial data on  $t = 0$  with the form of a short wavelength wave packet,  $a(x) e^{ix/\epsilon}$ , localized near a point  $p$ . The solution will launch wave packets along each of two characteristic curves. The envelopes are computed from those of the initial data, as in §5.2, and can take any form. One can send essentially arbitrary amplitude modulated signals.

The infinite variety of wave forms makes hyperbolic equations the preferred mode for communicating information, for example in hearing, sight, television, and radio. The model equations for the first are the linearized compressible inviscid fluid dynamics, a.k.a. acoustics. For the latter three it is Maxwell's equations. The telecommunication examples have the property that there is propagation with very small losses over large distances. The examples of wave packets and long distances show the importance of short wavelength and large time asymptotic analyses.

---

<sup>1</sup>The form is nondegenerate when its defining symmetric matrix is invertible.

Well posed Cauchy problems with finite speed lead to hyperbolic equations.<sup>2</sup> Since the fundamental laws of physics must respect the principles of relativity, finite speed is required. This together with causality requires hyperbolicity. Thus there are many equations from physics. Those which are most fundamental tend to have close relationships with Lorentzian geometry. D'Alembert's wave equation and the Maxwell equations are two examples. Problems with origins in general relativity are of increasing interest in the mathematical community, and it is the hope of hyperbolicians that the wealth of geometric applications of elliptic equations in Riemannian geometry will one day be paralleled by Lorentzian cousins of hyperbolic type.

A source of countless mathematical and technological problems of hyperbolic type are the equations of inviscid compressible fluid dynamics. Linearization of those equations yields linear acoustics. It is common that viscous forces are important only near boundaries, and therefore for many phenomena inviscid theories suffice. Inviscid models are often easier to compute numerically. This is easily understood as a small viscous term  $\epsilon^2 \partial^2 / \partial x^2$  introduces a length scale  $\sim \epsilon$ , and accurate numerics require a discretization small enough to resolve this scale, say  $\sim \epsilon/10$ . In dimensions  $1+d$  discretization of a unit volume for times of order 1 on such a scale requires  $10^4 \epsilon^{-4}$  mesh points. For  $\epsilon$  only modestly small, this drives computations beyond the practical. Faced with this, one can employ meshes which are only locally fine or try to construct numerical schemes which resolve features on longer scales without resolving the short scale structures. Alternatively, one can use asymptotic methods like those in this book to describe the boundary layers where the viscosity cannot be neglected (see for example [Grenier and Guès, 1998] or [Gérard-Varet, 2003]). All of these are active areas of research.

One of the key features of inviscid fluid dynamics is that smooth large solutions often break down in finite time. The continuation of such solutions as nonsmooth solutions containing shock waves satisfying suitable conditions (often called entropy conditions) is an important subarea of hyperbolic theory which is not treated in this book. The interested reader is referred to the conservation law references cited earlier. An interesting counterpoint is that for suitably dispersive equations in high dimensions, small smooth data yield global smooth (hence shock free) solutions (see §6.7).

The subject of geometric optics is a major theme of this book. The subject begins with the earliest understanding of the propagation of light. Observation of sunbeams streaming through a partial break in clouds or a

---

<sup>2</sup>See [Lax, 2006] for a proof in the constant coefficient linear case. The necessity of hyperbolicity in the variable coefficient case dates to [Lax, Duke J., 1957] for real analytic coefficients. The smooth coefficient case is due to Mizohata and is discussed in his book.



flashlight beam in a dusty room gives the impression that light travels in straight lines. At mirrors the lines reflect with the usual law of equal angles of incidence and reflection. Passing from air to water the lines are bent. These phenomena are described by the three fundamental principles of a physical theory called *geometric optics*. They are rectilinear propagation and the laws of reflection and refraction.

All three phenomena are explained by Fermat's principle of least time. The rays are locally paths of least time. Refraction at an interface is explained by positing that light travels at different speeds in the two media. This description is purely geometrical involving only broken rays and times of transit. The appearance of a minimum principle had important philosophical impact, since it was consistent with a world view holding that nature acts in a best possible way. Fermat's principle was enunciated twenty years before Römer demonstrated the finiteness of the speed of light based on observations of the moons of Jupiter.

Today light is understood as an electromagnetic phenomenon. It is described by the time evolution of electromagnetic fields, which are solutions of a system of partial differential equations. When quantum effects are important, this theory must be quantized. A mathematically solid foundation for the quantization of the electromagnetic field in  $1 + 3$  dimensional space time has not yet been found.

The reason that a field theory involving partial differential equations can be replaced by a geometric theory involving rays is that visible light has very short wavelength compared to the size of human sensory organs and common physical objects. Thus, much observational data involving light occurs in an asymptotic regime of very short wavelength. The short wavelength asymptotic study of systems of partial differential equations often involves significant simplifications. In particular there are often good descriptions involving rays. We will use the phrase *geometric optics* to be synonymous with short wavelength asymptotic analysis of solutions of systems of partial differential equations.

In optical phenomena, not only is the wavelength short but the wave trains are long. The study of structures which have short wavelength and are in addition very short, say a short pulse, also yields a geometric theory. Long wave trains have a longer time to allow nonlinear interactions which makes nonlinear effects more important. Long propagation distances also increase the importance of nonlinear effects. An extreme example is the propagation of light across the ocean in optical fibers. The nonlinear effects are very weak, but over 5000 kilometers, the cumulative effects can be large. To control signal degradation in such fibers, the signal is treated about every 30 kilometers. Still, there is free propagation for 30 kilometers which

needs to be understood. This poses serious analytic, computational, and engineering challenges.

A second way to bring nonlinear effects to the fore is to increase the amplitude of disturbances. It was only with the advent of the laser that sufficiently intense optical fields were produced so that nonlinear effects are routinely observed. The conclusion is that for nonlinearity to be important, either the fields or the propagation distances must be large. For the latter, dissipative losses must be small.

The ray description as a simplification of the Maxwell equations is analogous to the fact that classical mechanics gives a good approximation to solutions of the Schrödinger equation of quantum mechanics. The associated ideas are called the *quasiclassical approximation*. The methods developed for hyperbolic equations also work for this important problem in quantum mechanics. A brief treatment is presented in §5.2.2. The role of rays in optics is played by the paths of classical mechanics. There is an important difference in the two cases. The Schrödinger equation has a small parameter, Planck's constant. The quasiclassical approximation is an approximation valid for small Planck's constant. The mathematical theory involves the limit as this constant tends to zero. The Maxwell equations apparently have a small parameter too, the inverse of the speed of light. One might guess that rays occur in a theory where this speed tends to infinity. This is not the case. For the Maxwell equations in a vacuum the small parameter that appears is the wavelength which is introduced via the initial data. It is not in the equation. The equations describing the dispersion of light when it interacts with matter do have a small parameter, the inverse of the resonant frequencies of the material, and the analysis involves data tuned to this frequency just as the quasiclassical limit involves data tuned to Planck's constant. Dispersion is one of my favorite topics. Interested readers are referred to the articles [Donnat and Rauch, 1997] (both) and [Rauch, 2007].

Short wavelength phenomena cannot simply be studied by numerical simulations. If one were to discretize a cubic meter of space with mesh size  $10^{-5}$  cm so as to have five mesh points per wavelength, there would be  $10^{21}$  data points in each time slice. Since this is nearly as large as the number of atoms per cubic centimeter, there is no chance for the memory of a computer to be sufficient to store enough data, let alone make calculations. Such brute force approaches are doomed to fail. A more intelligent approach would be to use radical local mesh refinement so that the fine mesh was used only when needed. Still, this falls far outside the bounds of present computing power. Asymptotic analysis offers an alternative approach that is not only powerful but is mathematically elegant. In the scientific literature it is also embraced because the resulting equations sometimes have exact

solutions and scientists are well versed in understanding phenomena from small families of exact solutions.

Short wavelength asymptotics can be used to great advantage in many disparate domains. They explain and extend the basic rules of linear geometric optics. They explain the dispersion and diffraction of linear electromagnetic waves. There are nonlinear optical effects, generation of harmonics, rotation of the axis of elliptical polarization, and self-focusing, which are also well described.

Geometric optics has many applications within the subject of partial differential equations. They play a key role in the problem of solvability of linear equations via results on propagation of singularities as presented in §5.5. They are used in deriving necessary conditions, for example for hypoellipticity and hyperbolicity. They are used by Ralston to prove necessity in the conjecture of Lax and Phillips on local decay. Via propagation of singularities they also play a central role in the proof of sufficiency. Propagation of singularities plays a central role in problems of observability and controlability (see §5.6). The microlocal elliptic regularity theorem and the propagation of singularities for symmetric hyperbolic operators of constant multiplicity is treated in this book. These are the two basic results of linear microlocal analysis. These notes are not a systematic introduction to that subject, but they present an important part *en passant*.

Chapters 9 and 10 are devoted to the phenomenon of *resonance* whereby waves with distinct phases interact nonlinearly. They are preparatory for Chapter 11. That chapter constructs a family of solutions of the compressible  $2d$  Euler equations exhibiting three incoming wave packets interacting to generate an infinite number of oscillatory wave packets whose velocities are dense in the unit circle.

Because of the central role played by rays and characteristic hypersurfaces, the analysis of conormal waves is closely related to geometric optics. The reader is referred to the treatment of progressing waves in [Lax, 2006] and to [Beals, 1989] for this material.

**Acknowledgments.** I have been studying hyperbolic partial differential equations for more than forty years. During that period, I have had the pleasure and privilege to work for extended periods with (in order of appearance) M. Taylor, M. Reed, C. Bardos, G. Métivier, G. Lebeau, J.-L. Joly, and L. Halpern. I thank them all for the things that they have taught me and the good times spent together. My work in geometric optics is mostly joint with J.-L. Joly and G. Métivier. This collaboration is the motivation and central theme of the book. I gratefully acknowledge my indebtedness to them.

My research was partially supported by a sequence of National Science Foundation grants; NSF-DMS-9203413, 9803296, 0104096, and 0405899. The early stages of the research on geometric optics were supported by the Office of Naval Research under grant OD-G-N0014-92-J-1245. My visits to Paris have been supported by the CNRS, the Fondation Mathématiques de Paris, and the Institut Henri Poincaré. My visits to Pisa have been supported by the the Università di Pisa, INDAM, GNAMPA, and the Centro di Ricerca Matematica Ennio De Giorgi. Invitations as visiting professor at many departments (and notably l'Université de Paris Nord, l'École Normale Supérieure, and l'École Polytechnique for multiple invitations) have been crucially important in my research. I sincerely thank all these organizations and also the individuals responsible.



---

# Bibliography

- M. Alber, G. Luther, J. Marsden and J. Robbins, *Geometric phases, reduction and Lie–Poisson structure for the resonant three wave interaction*, *Physica D* **123** (1998), 271–290.
- S. Alinhac, *Équations différentielles: étude asymptotique; applications aux équations aux dérivées partielles* (French). [Differential equations: asymptotic study; applications to partial differential equations.] *Publications Mathématiques d’Orsay*, 80, 4. Université de Paris–Sud, Département de Mathématique, Orsay, 1980.
- S. Alinhac, *Blowup for Nonlinear Hyperbolic Equations*, Birkhäuser, 1995.
- S. Alinhac and P. Gérard, *Pseudo-differential Operators and the Nash–Moser Theorem*. (Translated from the 1991 French original by Stephen S. Wilson.) *Graduate Studies in Mathematics*, 82. American Mathematical Society, Providence, RI, 2007.
- S. Banach, *Théorie des Opérations Linéaires*, reprint, Chelsea Publishing Co., New York, 1955.
- S. Basu, R. Pollack and M.-F. Roy, *Algorithms in Real Algebraic Geometry, Second ed.* *Algorithms and Computation in Mathematics*, **10**, Springer-Verlag, Berlin, 2006.
- M. Beals, *Propagation and Interaction of Singularities in Nonlinear Hyperbolic Problems*, Birkhäuser, Boston, 1989.
- R. Benedetti and J.-L. Risler, *Real Algebraic and Semialgebraic Sets*, *Actualités Mathématiques*, Hermann, 1990.
- S. Benzoni-Gavage and D. Serre, *Multidimensional Hyperbolic Partial Differential Equations. First-order Systems and Applications*. *Oxford Mathematical Monographs*. The Clarendon Press, Oxford University Press, Oxford, 2007.
- N. Bloembergen, *Nonlinear Optics*, Benjamin, New York, 1964.
- J.-M. Bony, *Calcul symbolique et propagation des singularités pour les équations aux dérivées nonlinéaires*, *Ann. Sci. Ecole Norm. Sup.* **14** (1981), 209–246.

- P. Brenner, *The Cauchy problem for symmetric hyperbolic systems in  $L_p$* , Math. Scand. **19** (1966), 27–37.
- P. Brenner and W. von Wahl, *Global classical solutions of nonlinear wave equations*, Math. Z. **176** (1981), 87–121.
- A. Bressan, *Hyperbolic Systems of Conservation Laws. The One-dimensional Cauchy Problem*. Oxford Lecture Series in Mathematics and Its Applications, 20. Oxford University Press, Oxford, 2000.
- M. D. Bronšteĭn, *The Cauchy problem for hyperbolic operators with characteristics of variable multiplicity*. (Russian) Trudy Moskov. Mat. Obshch. **41** (1980), 83–99.
- A. Cauchy, *Mémoire sur l'intégration des équations différentielles*. Exercices d'analyse et de physique mathématiques, vol. 1, Paris, 1840.
- J. Chazarain and A. Piriou, *Introduction to the Theory of Linear Partial Differential Equations*. (Translated from the French.) Studies in Mathematics and Its Applications, 14. North-Holland Publishing Co., Amsterdam, 1982.
- J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, *Mathematical Geophysics. An Introduction to Rotating Fluids and the Navier–Stokes Equations*. Oxford Lecture Series in Mathematics and Its Applications, 32. The Clarendon Press, Oxford University Press, Oxford, 2006.
- Y. Choquet-Bruhat, *Ondes asymptotiques et approchées pour les systèmes d'équations aux dérivées partielles nonlinéaires*, J. Math. Pures. Appl. **48** (1969), 117–158.
- E. Conway and J. Smoller, *Global solutions of the Cauchy problem for quasi-linear first-order equations in several space variables*. Comm. Pure Appl. Math. **19** (1966), 95–105.
- R. Courant and D. Hilbert, *Methods of Mathematical Physics. Vol. I*. Interscience Publishers, Inc., New York, 1953.
- R. Courant and D. Hilbert, *Methods of Mathematical Physics. Vol. II*. (Vol. II by R. Courant.) Interscience Publishers (a division of John Wiley & Sons), New York, 1962.
- R. Courant, K. O. Friedrichs, and H. Lewy, *Über die partiellen differenzgleichungen der physik*, Math. Ann. **100** (1928–1929), 32–74.
- C. M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, Third edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 325. Springer-Verlag, Berlin, 2010.
- P. Donnat, “Quelque contributions mathématiques in optique nonlinéaire”, Ph.D. Thesis, Ecole Polytechnique, Paris, 1994.
- P. Donnat, J. L. Joly, G. Métivier and J. Rauch, *Diffraction nonlinear geometric optics*. Séminaire Equations aux Dérivées Partielles, Ecole Polytechnique, Palaiseau, 1995–1996.
- P. Donnat and J. Rauch, *Modelling the dispersion of light*, Singularities and Oscillations, (J. Rauch and M. Taylor, eds.), 17–35, IMA Volumes in Mathematics and Its Applications, 91, Springer-Verlag, New York, 1997.

- P. Donnat and J. Rauch, *Dispersive nonlinear geometric optics*, J. Math. Phys. **38**(3) (1997), 1484–1523.
- J. J. Duistermaat, *Fourier Integral Operators*. Courant Institute of Mathematical Sciences, New York University, New York, 1973.
- L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998.
- L. C. Evans and M. Zworski, *Semiclassical Analysis*, book manuscript available at Zworski's home page <http://math.berkeley.edu/~zworski/>, 2010.
- R. Feynman, R. Leighton and M. Sands, *The Feynman Lectures on Physics*, Addison-Wesley, Redwood City, CA, 1964.
- K. O. Friedrichs, *Symmetric hyperbolic linear differential equations*, Comm. Pure Appl. Math. **7** (1954), 345–392.
- A. Gabor, *Remarks on the wave front set of distribution*, Trans. AMS **170** (1972), 239–244.
- P. Garabedian, *Partial Differential Equations*, John Wiley & Sons, Inc., New York-London-Sydney, 1964.
- L. Gårding, *Linear hyperbolic partial differential equations with constant coefficients*. Acta Math. **85** (1951), 1–62.
- V. Georgiev, S. Lucente and G. Zillotti, *Decay estimates for hyperbolic systems*, Hokkaido Math. J. **33** (2004), 83–113.
- D. Gérard-Varet, *A geometric optics type approach to fluid boundary layers*, Comm. Part. Diff. Eq., **28** (2003), 1605–1626.
- J. Ginibre and G. Velo, *Generalized Strichartz inequalities for the wave equation*. J. Funct. Anal. **133** (1995), 50–68.
- J. Ginibre and G. Velo, *The global Cauchy problem for the nonlinear Klein-Gordon equation*. Math. Z. **189** (1985), 487–505.
- J. Glimm and P. Lax, *Decay of solutions of systems of nonlinear hyperbolic conservation laws*, Memoirs of the American Mathematical Society, No. 101, American Mathematical Society, Providence, RI, 1970.
- E. Grenier and O. Guès, *Boundary layers for viscous perturbations of noncharacteristic quasilinear hyperbolic problems*, J. Diff. Eq. **143** (1998) 110–146.
- O. Guès, *Ondes multidimensionnelles epsilon stratifiées et oscillations*, Duke Math. J. **68** (1992), 401–446.
- O. Guès, *Développement asymptotique de solutions exactes de systèmes hyperboliques quasilinéaires*, Asympt. Anal. **6** (1993), 241–269.
- O. Guès and J. Rauch, *Hyperbolic  $L^p$  multipliers are translations*, Comm. PDE **31** (2006), 431–443.
- V. Guillemin and A. Pollack, *Differential Topology*, Prentice Hall, Inc., Englewood Cliffs, NJ, 1974.
- A. Haar, *Über eindeutigkeit un analytizität der lösungen parieler differenzialgleichen*, Atti del Congr. Intern. dei Mat., Bologna, vol. **3** (1928), 5–10.



- J. Hadamard, *Lectures on the Cauchy Problem in Linear Partial Differential Equations*, reissue of 1923 original, Dover Publ. New York, 2003.
- L. Hörmander, *Fourier integral operators I*, Acta Math. **127** (1971), 79–183.
- L. Hörmander, *The Analysis of Linear Partial Differential Operators vols. I, II*, Springer-Verlag, Berlin, 1983.
- L. Hörmander, *Lectures on Nonlinear Hyperbolic Differential Equations*, Springer-Verlag, Berlin-New York-Heidelberg, 1997.
- J. Hunter and J. Keller, *Weakly nonlinear high frequency waves*, Comm. Pure Appl. Math. **36** (1983) no. 5, 547–569.
- J. Hunter and J. Keller, *Caustics of nonlinear waves*. Wave Motion **9** (1987) no. 5, 429–443.
- J. Hunter, A. Majda and R. Rosales, *Resonantly interacting weakly nonlinear hyperbolic waves II*, Stud. Appl. Math. **75** (1986), 187–226.
- N. Iwasaki, *Local decay of solutions for symmetric hyperbolic systems with dissipative and coercive boundary conditions in exterior domains*, Publ. RIMS Kyoto Univ. **5** (1969), 193–218.
- H. K. Jenssen, *Blowup for systems of conservation laws*, SIAM J. Math. Anal. **31** (2000), no. 4, 894–908.
- F. John, *Partial Differential Equations*, 4<sup>th</sup> ed., Springer-Verlag, New York, 1982.
- J. L. Joly, *Sur la propagation des oscillations par un système hyperbolique semi-linéaire en dimension 1 d'espace*. (French) [On the propagation of oscillations by a hyperbolic semilinear system in one space dimension.] C. R. Acad. Sci. Paris Sér. I Math. **296** (1983), no. 15, 669–672.
- J. L. Joly, G. Métivier and J. Rauch, *Formal and rigorous nonlinear high frequency hyperbolic waves*, in Nonlinear Hyperbolic Waves and Field Theory, (M. K. Murthy and S. Spagnolo eds.), Pitman Research Notes in Math, **253** (1992), 121–143.
- J. L. Joly, G. Métivier and J. Rauch, *Resonant one dimensional nonlinear geometric optics*, J. Funct. Anal. **114** (1993), 106–231.
- J. L. Joly, G. Métivier and J. Rauch, *Generic rigorous asymptotic expansions for weakly nonlinear multidimensional oscillatory waves*, Duke Math. J. **70** (1993), 373–404.
- J. L. Joly, G. Métivier and J. Rauch, *Coherent nonlinear waves and the Wiener algebra*, Ann. Inst. Fourier, **44** (1994), 167–196.
- J. L. Joly, G. Métivier and J. Rauch, *A nonlinear instability for  $3 \times 3$  systems of conservation laws*, Comm. Math. Phys. **162** (1994), 47–59.
- J. L. Joly, G. Métivier and J. Rauch, *Dense oscillations for the compressible 2-d Euler equations*. *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. XIII (Paris, 1994/1996)*, 134–166, Pitman Res. Notes Math. Ser., 391, Langman, Harlow, 1998.
- J. L. Joly, G. Métivier and J. Rauch, *Dense oscillations for the Euler equations II, Hyperbolic problems: theory, numerics, applications*, (Stony Brook, NY, 1994), 425–430, World Sci. Publ., River Edge, NJ, 1996.

- J. L. Joly, G. Métivier and J. Rauch, *Diffractive nonlinear geometric optics with rectification*, Indiana U. Math. J. **47** (1998), 1167–1242.
- J. L. Joly, G. Métivier and J. Rauch, *Dense oscillations for the compressible two dimensional Euler equations*. Nonlinear Partial Differential Equations and their Applications, College de France Seminar 1992–1993, (D. Cioranescu and J. L. Lions, eds.), Pitman Research Notes in Math. **391**, Longman Publs. 1998.
- J. L. Joly, G. Métivier and J. Rauch, *Hyperbolic domains of dependence and Hamilton–Jacobi equations*, J. Hyp. Diff. Eq., **2** (2005), 713–744.
- J. L. Joly and J. Rauch, *Justification of multidimensional single phase semilinear geometric optics*, Trans. AMS **330** (1992), 599–623.
- K. Kajitani and A. Satoh, *Time decay estimates for linear symmetric hyperbolic systems with variable coefficients and its applications*. Phase Space Analysis of Partial Differential Equations, (F. Colombini and L. Pernazza, eds.), Centro De Giorgi, Scuola Normale Superiore, Pisa, 2004.
- M. Keel and T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math. **120** (1998), 955–980.
- J. Keller, *On solutions of nonlinear wave equations*, Comm. Pure Appl. Math. **10** (1957), 523–530.
- S. Klainerman, *On the work and legacy of Fritz John, 1934–1991. Dedicated to the memory of Fritz John*. Comm. Pure Appl. Math. **51** (1998), no. 9–10, 991–1017.
- H. O. Kreiss, *Über sachgemässe Cauchyprobleme*, Math. Scand. **13** (1963), 109–128.
- S. Kruzhkov, *First order quasilinear equations with several space variables*, Math. USSR, Sbornik **10** (1970), 217–243.
- P. D. Lax, *On Cauchy’s problem for hyperbolic equations and the differentiability of solutions of elliptic equations*, Comm. Pure Appl. Math. **8** (1955), 615–633.
- P. D. Lax, *Hyperbolic systems of conservation laws II*, Comm. Pure. Appl. Math. **10** (1957), 537–566.
- P. D. Lax, *Asymptotic solutions of oscillatory initial value problems*, Duke Math. J. **24** (1957), 627–646.
- P. D. Lax, *Lectures on Hyperbolic Partial Differential Equations*, Stanford University Lecture Notes, 1963.
- P. D. Lax, *Shock waves and entropy*, Contributions to Nonlinear Functional Analysis, (E. Zarantonello, ed.), Academic Press, New York, 1971.
- P. D. Lax, *Hyperbolic Partial Differential Equations*, Courant Lecture Notes, 14, American Mathematical Society, Providence, RI, 2006.
- G. Lebeau, *Nonlinear optic and supercritical wave equation, Hommage à Pascal Laubin*. Bull. Soc. Roy. Sci. Liège **70** (2001), no. 4–6, 267–306.
- G. Lebeau, *Perte de régularité pour les équations d’ondes sur-critiques*, Bull. Soc. Math. France **133** (2005) no. 1, 145–157.
- J. Leray, *Lectures on Hyperbolic Partial Differential Equations*, Institute for Advanced Study, Princeton, 1953.
- H. Levy, *An example of a smooth linear partial differential equation without solution*, Ann. of Math. **66** (1957), 155–158.

- S. Lucente and G. Zillotti, *A decay estimate for a class of hyperbolic pseudo-differential equations*, Math. App. **10** (1999), 173–190, Atti, Acc. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9).
- D. Ludwig, *Exact and asymptotic solutions of the Cauchy problem*, Comm. Pure Appl. Math. **13** (1960), 473–508.
- D. Ludwig, *Conical refraction in crystal optics and hydromagnetics*, Comm. Pure Appl. Math. **14** (1961), 113–124.
- R. K. Luneburg, *The Mathematical Theory of Optics*, Brown Univ. Press, Providence, RI, 1944.
- A. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*. Applied Mathematical Sciences, 53. Springer-Verlag, New York, 1984.
- A. Majda, *Nonlinear geometric optics for hyperbolic systems of conservation laws*, Oscillations Theory, Computation, and Methods of Compensated Compactness (C. Dafermos, J. Ericksen, D. Kinderlehrer, and I. Muller, eds.), IMA Volumes on Mathematics and its Applications vol. 2 pp. 115–165, Springer-Verlag, New York, 1986.
- A. Majda and R. Rosales, *Resonantly interacting weakly nonlinear hyperbolic waves*, Stud. Appl. Math. **71** (1986), 149–179.
- A. Majda, R. Rosales, and M. Schonbek, *A canonical system of integro-differential equations in nonlinear acoustics*, Stud. Appl. Math. **79** (1988), 205–262.
- D. McLaughlin, G. Papanicolaou and L. Tartar, *Weak limits of semilinear hyperbolic systems with oscillating data*, Macroscopic Modeling of Turbulent Flows, Lecture Notes in Physics, 230, Springer-Verlag, 1985, pp. 277–298.
- G. Métivier, *The Mathematics of Nonlinear Optics*, Handbook of differential equations: evolutionary equations. vol. **V**, Elsevier/North-Holland, Amsterdam, 2009, pp. 169–313.
- G. Métivier, *Problèmes de Cauchy et ondes non linéaires*. (French) [Cauchy problems and nonlinear waves] Journées “Équations aux dérivées partielles” (Saint Jean de Monts, 1986), No. I, École Polytechnique, Palaiseau, 1986.
- G. Métivier, *Para-differential Calculus and Applications to the Cauchy problem for Nonlinear Systems*. Centro di Ricerca Matematica Ennio De Giorgi (CRM) Series, 5. Edizioni della Normale, Pisa, 2008.
- G. Métivier and J. Rauch, *Real and complex regularity are equivalent for hyperbolic characteristic varieties*, Differential Integral Equations, **16** (2003), 993–999.
- G. Métivier and J. Rauch, *Invariance and stability of the profile equations of geometric optics*, Acta Math. Sci. **31** B6 (2011), 2141–2158.
- Y. Meyer, *Remarque sur un théoreme de J.-M. Bony*, Rend. Circ. Mat. Palermo, II. Ser. 1981, Suppl. **1** (1981) 1–20.
- S. Mizohata, *Lectures on the Cauchy Problem*, Notes by M. K. V. Murthy and B. V. Singbal. Tata Institute of Fundamental Research Lectures on Mathematics and Physics, No. 35, Tata Institute of Fundamental Research, Bombay, 1965.

- T. S. Motzkin and O. Tausky, *Pairs of matrices with property L*, Trans. AMS **73** (1952), 108–114.
- G. Peano, *Sull'inegrabilità della equazioni differenziali di primo ordine*, Atti. Acad. Torino, **21A** (1886), 677–685.
- J. Ralston, *Solutions of the wave equation with localized energy*, Comm. Pure Appl. Math. **22** (1969) 807–823.
- J. Rauch, *An  $L^2$  proof that  $H^s$  is invariant under nonlinear maps for  $s > n/2$* . Global Analysis - Analysis on Manifolds (T. Rassias, ed.), Teubner Texte zur Math., band 57, Leipzig, 1983.
- J. Rauch, *Partial Differential Equations*, Graduate Texts in Math., 128, Springer-Verlag, New York, 1991.
- J. Rauch, *Lectures on geometric optics*. Nonlinear Wave Phenomena (L. Caffarelli and W. E. eds.), IAS/Park City Math. Ser., 5, American Mathematical Society, Providence, RI, 1998.
- J. Rauch, *Precise finite speed with bare hands*. Methods Appl. Anal. **12** (2005), no. 3, 267–277.
- J. Rauch, *Á Travers un Prism*. Lecons de Mathématiques d'Aujourd'hui, vol. III (E. Charpentier and N. Nikolski, eds.), Cassini Pubs., Paris, 2007, pp. 35–68.
- J. Rauch, *Precise finite speed and uniqueness in the Cauchy problem for symmetrizable hyperbolic systems*, Trans. AMS, **363** no. 3 (2011), 1161–1182.
- J. Rauch and F. Massey, *Differentiability of solutions to hyperbolic initial-boundary value problems*, Trans. AMS, **189** (1974), 303–318.
- J. Rauch and M. Reed, *Nonlinear microlocal analysis of semilinear hyperbolic systems in one space dimension*, Duke Math. J. **49** (1982), 379–475.
- J. Rauch and M. Reed, *Bounded, stratified, and striated solutions of hyperbolic systems*. Nonlinear Partial Differential Equations and Their Applications Vol. IX (H. Brezis and J. L. Lions, eds.), Pitman Research Notes in Math., **181** (1989), 334–351.
- J. Rauch and M. Taylor, *Exponential decay of solutions to hyperbolic equations in bounded domains*, Indiana U. Math. J. **24** (1974), 79–86.
- M. Reed and B. Simon, *Scattering Theory, Methods of Mathematical Physics, Vol. III*, Academic Press, 1983.
- R. Sakamoto, *Hyperbolic Boundary Value Problems*, Translated from the Japanese by Katsumi Miyahara, Cambridge University Press, Cambridge-New York, 1982.
- A. Satoh, *Scattering for nonlinear symmetric hyperbolic systems*, Tsukuba J. Math. **29** (2005), no. 2, 285–307
- J. Schauder, *Das anfangswertprobleme einer quasilinearen hyperbolischen differentialgleichung zweiter ordnung*, Fund. Math. **24** (1935), 213–246.
- S. Schochet, *Fast singular limits of hyperbolic equations*, J. Diff. Eq. **114** (1994), 476–512.

- D. Serre, *Systems of Conservation Laws. 1. Hyperbolicity, Entropies, Shock Waves*. (Translated from the 1996 French original by I. N. Sneddon.) Cambridge University Press, Cambridge, 1999.
- D. Serre, *Systems of Conservation Laws. 2. Geometric Structures, Oscillations, and Initial-boundary Value Problems*. (Translated from the 1996 French original by I. N. Sneddon.) Cambridge University Press, Cambridge, 2000.
- J. Shatah and M. Struwe, *Geometric Wave Equations*, Courant Lecture Notes in Mathematics, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1998.
- J. Smoller, *Shock Waves and Reaction-Diffusion Equations*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 258. Springer-Verlag, Berlin, 1983.
- C. Sogge, *Lectures on nonlinear wave equations*, Second Edition. International Press, Boston, MA, 2008.
- E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, 1970.
- E. Stein, *Harmonic Analysis, Real Variable Methods, Orthogonality, and Oscillatory integrals*, with the assistance of T. Murphy, Princeton Mathematical Series, 43, Princeton University Press, 1993.
- W. Strauss, *Nonlinear Wave Equations*, CBMS Regional Conf. Series, 73, American Mathematical Society, Providence, RI, 1989.
- T. Tao, *Nonlinear Dispersive Equations. Local and Global Analysis*. CBMS Regional Conference Series in Mathematics, 106, American Mathematical Society, Providence, RI, 2006.
- M. Taylor, *Pseudodifferential Operators*, Princeton Math. Series, 34, Princeton University Press, Princeton, 1981.
- M. E. Taylor, *Pseudodifferential operators and nonlinear PDE*. Progress in Mathematics, 100. Birkhäuser Boston Inc., Boston, 1991.
- M. Taylor, *Partial Differential Equations, Basic Theory*, Texts in Applied Math., 23, Springer-Verlag, New York, 1996.
- M. Taylor, *Partial Differential Equations. II. Qualitative Studies of Linear Equations*, Applied Mathematical Sciences, 116. Springer-Verlag, New York, 1996.
- M. Taylor, *Partial Differential equations. III. Nonlinear equations*, Applied Mathematical Sciences, 117. Springer-Verlag, New York, 1997.
- T. Wagenmaker, “Analytic solutions and resonant solutions of hyperbolic partial differential equations”, Ph.D. Thesis, University of Michigan, 1993.
- G. Whitham, *Linear and Nonlinear Waves*, Wiley-Interscience, New York, 1974.
- R. Young, *Exact solutions to degenerate conservation laws*, SIAM J. Math. Anal. **30** (1999), no. 3, 537–558.
- R. Young, *Blowup of solutions and boundary instabilities in nonlinear hyperbolic equations*, Commun. Math. Sci. **1** (2003), no. 2, 269–292.
- V. Yudovich, *Nonstationary flows of an ideal incompressible fluid*, Zh. Vycg. Math. **3** (1963), 1032–1066.

---

# Index

- $Q(Y)$ , partial inverse, 117  
 $\mathbb{A}$ , Wiener algebra, 96  
 $\mathcal{E}$ , restriction of  $\mathbf{E}$  to quasiperiodic profiles, 307  
 $E_\alpha(\xi)$ , spectral projection for  $\alpha^{\text{th}}$  sheet, 95  
 $\mathbf{E}$ , 267–275  
 $\mathbf{E}$ , projection onto kernel for trigonometric series, 306  
 $\mathcal{F}$ , Fourier transform, 45  
 $H^s(\mathbb{R}^d)$ , Sobolev space, 46  
 $H_\epsilon^s$ ,  $\epsilon\partial$  Sobolev space, 278–282  
 $H^s(y)$ , local Sobolev space, 134  
 $H^s(y, \eta)$ , microlocal Sobolev space, 137  
 $L_t^q L_x^r$ , Strichartz norms, 111  
 $\text{Op}(\sigma, \partial_x)$ , order  $\sigma$  differential operators, 50  
 $PC^k$ , piecewise smooth functions, 11  
 $\mathcal{Q}$ , restriction of  $\mathbf{Q}$  to quasiperiodic profiles, 307  
 $Q(y)$ , partial inverse of  $L_1(y, d\phi(y))$ , 155  
 $Q(\alpha)$ , partial inverse of  $L_1(\alpha)$ , 306  
 $\mathbf{Q}$ , partial inverse for trigonometric series, 307  
 $S^m(\Omega \times \mathbb{R}^N)$ , classical symbols, 181  
 $WF$ , wavefront set, 139  
 $WF_s$ ,  $H^s$  wavefront set, 137  
 $x\xi$ , 19  
 $\pi(y, \eta)$ , spectral projection on  $\ker L_1(y, \eta)$ , 71  
 $\sigma$ -admissible, for Strichartz inequality, 112  
 $\Omega_t$ , 271  
 $\Omega(t)$ , 271  
bicharacteristic  
  and Hamilton-Jacobi theory, 210  
  and propagation of singularities, 188–195  
  and stabilization, 201–205  
  and transport, 151, 165–166  
Borel’s theorem, 128, 151, 229, 270, 313  
boundary layer, xv  
breakdown/blow up, 224–227, 237–238, 325, 330  
Brenner’s theorem, xii, 12, 102–104  
Burgers’ equation  
  breakdown, 237  
  dependence on initial data, 241  
  Liouville’s theorem, 238  
  method of characteristics, 237  
  time of nonlinear interaction, 303  
Cauchy problem, xiv  
  fully nonlinear scalar, 206–214  
  linear, 43–90  
  nonlinear, 215–257  
  quasilinear, 230–242  
  small data, 242–246  
  subcritical, 246–257  
characteristic curves, 2, 13  
  and breakdown, 237–238  
  and finite speed, 58–59  
  fully nonlinear scalar, 206–214  
  method of, 2–16

- characteristic polynomial, 65  
   Euler's equations, 334  
   Maxwell's equations, 67, 72  
 characteristic variety, 65, 163  
   curved and flat, 95  
 commutator, 8, 233–235, 281–282  
   **E**, 271  
 computer approximations, xv, xvii, 166  
 conservation of charge, 44  
 constant rank hypothesis, 154–155  
 continuity equation, 44  
 controllability, xviii, 195  
 corrector, 26, 126, 310  
 curved sheets, of characteristic variety, 95  
  
 D'Alembert's formula, 3, 13, 253  
 diffractive geometric optics, xii, 25  
 dimensional analysis, 221  
 dispersion, 92, 111  
   dispersive behavior, 91–117, 149  
   relation, 14, 23, 34, 36, 78, 92  
   Schrödinger equation, 150  
 dispersive geometric optics, xii, xvii, 261  
 domains of influence and determinacy, 58–59, 70–71  
 Duhamel's formula, 57  
  
 eikonal equation  
   and constant rank hypothesis, 178  
   and Lax parametrix, 177  
   and three wave interaction, 292–294  
   for nonlinear geometric optics, 266  
   Hamilton-Jacobi theory for, 206–214  
   Schrödinger equation, 150–151  
   simple examples, 152–155  
 elliptic operator, xiv, 79  
 elliptic regularity theorem, 132–140  
   elliptic case, 6  
   microlocal, 136–140  
 emission, 79–83  
 energy  
   conservation of, 13, 14, 45, 147, 169–177, 246  
   method, 29–30, 36–38  
 energy, conservation of, 78  
 Euler equations  
   compressible inviscid, 310  
   dense oscillations for, 333–350  
  
 Fermat's principle of least time, xvi  
 finite speed, xiv, 10, 29, 38, 58–83  
  
   for semilinear equations, 224  
   speed of sound, 341  
 flat parts, of characteristic variety, 95  
 Fourier integral operator, xii, 177, 180–188  
 Fourier transform, definition, 45  
 frequency conversion, 302  
 fundamental solution, 13–15, 20  
  
 geometric optics, xi, xv, xvi  
   cautionary example, 27  
   elliptic, 123–132  
   from solution by Fourier transform, 20–27  
   linear hyperbolic, 141–177  
   nonlinear multiphase, 291–350  
   nonlinear one phase, 259, 277–289, 310  
   physical, 1, 16  
   second order scalar, 143–149  
 Gronwall's Lemma, 50  
 group velocity, 34, 36  
   and decay for maximally dispersive systems, 111  
   and Hamilton-Jacobi theory, 210  
   and nonstationary phase, 16–20  
   and smooth variety hypothesis, 163  
   conormal to characteristic variety, 71–78  
   for anisotropic wave equation, 27  
   for curved sheets of characteristic variety, 100  
   for D'Alembert's equation, 23  
   for scalar second order equations, 145–149  
   for Schrödinger's equation, 149  
 Guoy shift, 173  
  
 Haar's inequality, 7–12  
 Hamilton-Jacobi theory  
   for hyperbolic problems, 152–155  
   Schrödinger equation, 150–151  
 Hamilton-Jacobi theory, 206–214  
 harmonics, generation of, xviii, 260–263  
 homogeneous Sobolev norm, 112  
 hyperbolic  
   constant coefficient, 84–89  
   constant multiplicity, 89, 178  
   strictly, xiv, 6, 89  
   second order, 146  
   symmetric, 44–58, 88, 151–195  
   definition constant coefficients, 45

- definition variable coefficient, 47
- images, method of, 30–33
- inequality of stationary phase, 106
- influence curve, 79–83
- integration by parts, justification, 61
- inviscid compressible fluid dynamics, xv
- Keller’s blowup theorem, 249
- Klein–Gordon equation, 14, 19, 73, 78, 146, 246–257
- Kreiss matrix theorem, xii, 88
- lagrangian manifold, 213
- Lax parametrix, 177–195
- Liouville
  - Liouville number, 312
  - Liouville’s theorem, 238, 311
- Littlewood–Paley decomposition, 109, 115, 222, 254
- maximally dispersive, 92, 104, 242–246
- Maxwell’s equations, xiv
  - characteristic variety, 67
  - circular and elliptical polarization, 73
  - eikonal equation, 154
  - introduction, 44–46
  - plane waves, 72
  - propagation cone, 68
  - rotation of polarization, 288–289
  - self phase modulation, 287–288
- microlocal
  - analysis, xi, xviii
  - elliptic regularity theorem, xviii, 136–140
  - propagation of singularities theorem, 177–195
    - applied to stabilization, 195–205
- Moser’s inequality, 224, 325
  - $H_e^s$ , 284
- nondegenerate phase, 182
- nondispersive, 99
- nonstationary phase, xii
  - and Fourier integral operators, 180–188
  - and group velocity, 16–20
  - and resonance, 293, 324
  - and the stationary phase inequality, 121–122
- observability, xviii
- operator
  - pseudodifferential, xii, 136, 186
  - transposed, 134
- oscillations
  - creation of, 310
  - homogeneous, 302, 336–338
- oscillatory integrals, 180–188
- partial inverse
  - for a single phase, 155
  - multiphase
    - on quasiperiodic profiles, 307
    - on trigonometric series, 306
  - of a matrix, 117
- perturbation theory, 239
  - for semisimple eigenvalues, 117–119, 164
  - generation of harmonics, 262–263
  - quasilinear, 239
  - semilinear, 227–230
  - small oscillations, 259–262
- phase velocities, 74, 145
- piecewise smooth
  - definition  $d = 1$ , 10
  - function, wavefront set of, 140
  - solutions for refraction, 38
  - solutions in  $d = 1$ , 11
- plane wave, 17, 142–143
- polarization
  - in nonlinear geometric optics, 268
  - linear, circular, and elliptical, 73
  - of plane waves, 72
  - rotation of axis, xviii, 288
- polyhomogeneous, 182
- principal symbol, 64
- profile equations
  - quasilinear, 302–314
  - semilinear, 265–275, 314–315
- projection (a.k.a. averaging) operator
  - $\mathbf{E}$ , 267–275
- propagation cone, 64–71, 75
- propagation of singularities, xviii, 1
  - $d = 1$  and characteristics, 10–11
  - $d = 1$  and progressing waves, 12–16
  - using Fourier integral operators, 177–195
- pulse, *see* wave
- purely dispersive, 99–100
- quasiclassical limit of quantum
  - mechanics, xvii, 149–151, 160–161, 278



- ray cone, 77, 78
- ray tracing algorithms, 166
- rays, 144–145
  - and conormal waves, xviii
  - spread of, 25
  - transport along, 2, 23, 161–177, 285–289
  - tube of, 26, 34, 100, 147, 148, 288
- rectilinear propagation, xvi, 20–27
- reflection
  - coefficient of, 32, 35, 36
  - law of, xvi, 28
  - operator, 30
  - total, 42
- refraction, Snell's law, xvi, 36–42
- resonance, xviii
  - collinear, 341
  - examples of, 291–302, 317–332
  - introduction to, 291–294
  - quadratic, definition, 341
  - quasilinear, 302–314, 327–332
  - relation, 292
  - relations for Euler equations, 341–342
  - semilinear, 291–302, 314–315, 321–327
- Römer, xvi
- Schauder's lemma, 217–222
- Schrödinger's equation, 149–151
- self phase modulation, 287, 288
- semiclassical limit of quantum mechanics, 278
- semisimple eigenvalue, 117
- short wavelength asymptotic analysis, xvi
- singularities
  - (microlocal) of piecewise smooth functions, 140
  - and the method of characteristics, 10–11
  - for progressing waves, 12–16
  - propagation for piecewise smooth waves, 10–11, 194
  - propagation global in time, 192–195
  - propagation local in time, 188–191
  - propagation of, xiv, xviii
  - propagation of and stabilization, 195–205
- slowly varying envelope approximation, 287
- small divisor, 310–313
  - hypothesis, 312, 318
  - for Euler equations, 339
- smooth characteristic variety
  - hypothesis, 163–178, 286–289
- smooth points, of the characteristic variety, 76–78
- Snell's law, *see* refraction
- Sobolev embedding, 216, 220
- space like, 146
- spectral projection, 117
- spectrum
  - of a periodic function, 309
  - of  $F(V)$ , 315
  - of principal profile, 309
- stability
  - Hadamard's notion of, 43
  - theorem, quasilinear, 314
  - theorem, semilinear, 283
- stationary phase, 210
- stationary phase inequality, 120–122
- stationary point, nondegenerate, 120
- stratification theorem, 76, 94, 99
- Strichartz inequalities, xii, 111–117, 251–257
- three wave interaction
  - infinite system, 337
  - ode, 298–302, 330–332, 336
  - pde, 294–298, 322–327
  - resonance of order three, 341
- time of nonlinear interaction
  - quasilinear, 303
  - semilinear, 262
- time-like, 146
  - cone, 64–71
- transport equation, 144, 146
- wave
  - acoustic, xiv, 341
  - conormal, xviii, 16, 194
  - plane, 14, 33–34, 71–79, 150
  - progressing, 12–16
  - shock, xv, 12
  - short pulse, xvi
  - spherical, 32, 171–173
  - vorticity, 341
  - wave packet, xiv, 1, 22–28, 34–36, 142
  - wave train, 1, 291
- wave number, good and bad, 93–96
- wavefront set, 136–140, 186–195, 201–204
- Wiener algebra, 96

## WKB

- from Euler's method, 123–127
- from iterative improvement, 125–127
- from perturbation theory, 131–132
- from solution by Fourier transform,  
20–27

Young measure, 293



This book introduces graduate students and researchers in mathematics and the sciences to the multifaceted subject of the equations of hyperbolic type, which are used, in particular, to describe propagation of waves at finite speed.

Among the topics carefully presented in the book are nonlinear geometric optics, the asymptotic analysis of short wavelength solutions, and nonlinear interaction of such waves. Studied in detail are the damping of waves, resonance, dispersive decay, and solutions to the compressible Euler equations with dense oscillations created by resonant interactions. Many fundamental results are presented for the first time in a textbook format. In addition to dense oscillations, these include the treatment of precise speed of propagation and the existence and stability questions for the three wave interaction equations.

One of the strengths of this book is its careful motivation of ideas and proofs, showing how they evolve from related, simpler cases. This makes the book quite useful to both researchers and graduate students interested in hyperbolic partial differential equations. Numerous exercises encourage active participation of the reader.

The author is a professor of mathematics at the University of Michigan. A recognized expert in partial differential equations, he has made important contributions to the transformation of three areas of hyperbolic partial differential equations: nonlinear microlocal analysis, the control of waves, and nonlinear geometric optics.

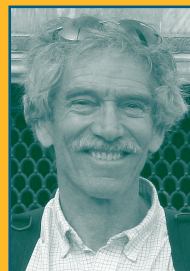


Photo by Geraldine Callisto Kaylor

ISBN 978-0-8218-7291-8



9 780821 872918

GSM/133



For additional information  
and updates on this book, visit

[www.ams.org/bookpages/gsm-133](http://www.ams.org/bookpages/gsm-133)

AMS on the Web  
[www.ams.org](http://www.ams.org)