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To Geraldine.

Contents

Preface

P.1. How this book came to be, and its peculiarities

This book presents an introduction to hyperbolic partial differential equations. A major subtheme is linear and nonlinear geometric optics. The two central results of linear microlocal analysis are derived from geometric optics. The treatment of nonlinear geometric optics gives an introduction to methods developed within the last twenty years, including a rethinking of the linear case.

Much of the material has grown out of courses that I have taught. The crucial step was a series of lectures on nonlinear geometric optics at the Institute for Advanced Study/Park City Mathematics Institute in July 1995. The Park City notes were prepared with the assistance of Markus Keel and appear in [Rauch, 1998]. They presented a straight line path to some theorems in nonlinear geometric optics. Graduate courses at the University of Michigan in 1993 and 2008 were important. Much of the material was refined in invited minicourses:

- École Normale Supérieure de Cachan, 1997;
- Nordic Conference on Conservation Laws at the Mittag-Leffler Institute and KTH in Stockholm, December 1997 (Chapters 9–11);
- Centro di Ricerca Matematica Ennio De Giorgi, Pisa, February 2004;
- Université de Provence, Marseille, March 2004 (§3.4, 5.4, Appendix 2.I);
- Università di Pisa, February–May 2005, March–April 2006 (Chapter 3, §6.7, 6.8), March–April 2007 (Chapters 9–11);
- Université de Paris Nord, February 2006–2010 (§1.4–1.7).

The auditors included many at the beginnings of their careers, and I would like to thank in particular R. Carles, E. Dumas, J. Bronski, J. Colliander, M. Keel, L. Miller, K. McLaughlin, R. McLaughlin, H. Zag, G. Crippa, A. Figalli, and N. Visciglia for many interesting questions and comments.

The book is aimed at the level of graduate students who have studied one hard course in partial differential equations. Following the lead of the book of Guillemin and Pollack (1974), there are exercises scattered throughout the text. The reader is encouraged to read with paper and pencil in hand, filling in and verifying as they go. There is a big difference between passive reading and active acquisition. In a classroom setting, correcting students' exercises offers the opportunity to teach the writing of mathematics.

To shorten the treatment and to avoid repetition with a solid partial differential equations course, basic material such as the fundamental solution of the wave equation in low dimensions is not presented. Naturally, I like the treatment of that material in my book *Partial Differential Equations* [Rauch, 1991].

The choice of subject matter is guided by several principles. By restricting to symmetric hyperbolic systems, the basic energy estimates come from integration by parts. The majority of examples from applications fall under this umbrella.

The treatment of constant coefficient problems does not follow the usual path of describing classes of operators for which the Cauchy problem is weakly well posed. Such results are described in Appendix 2.I along with the Kreiss matrix theorem. Rather, the Fourier transform is used to analyse the dispersive properties of constant coefficient symmetric hyperbolic equations including Brenner's theorem and Strichartz estimates.

Pseudodifferential operators are neither presented nor used. This is not because they are in any sense vile, but to get to the core without too many pauses to develop machinery. There are several good sources on pseudodifferential operators and the reader is encouraged to consult them to get alternate viewpoints on some of the material. In a sense, the expansions of geometric optics are a natural replacement for that machinery. Lax's parametrix and Hörmander's microlocal propagation of singularities theorem require the analysis of oscillatory integrals as in the theory of Fourier integral operators. The results require only the method of nonstationary phase and are included.

The topic of caustics and caustic crossing is not treated. The sharp linear results use more microlocal machinery and the nonlinear analogues are topics of current research. The same is true for supercritical nonlinear geometric optics which is not discussed. The subjects of dispersive and diffractive nonlinear geometric optics in contrast have reached a mature state. Readers of this book should be in a position to readily attack the papers describing that material.

The methods of geometric optics are presented as a way to understand the qualitative behavior of partial differential equations. Many examples proper to the theory of partial differential equations are discussed in the text, notably the basic results of microlocal analysis. In addition two long examples, stabilization of waves in §5.6 and dense oscillations for inviscid compressible fluid flow in Chapter 11 are presented. There are many important examples in science and technology. Readers are encouraged to study some of them by consulting the literature. In the scientific literature there will not be theorems. The results of this book turn many seemingly ad hoc approximate methods into rigorous asymptotic analyses.

Only a few of the many important hyperbolic systems arising in applications are discussed. I recommend the books [Courant, 1962], [Benzoni-Gavage and Serre, 2007], and [Métivier, 2009]. The asymptotic expansions of geometric optics explain the physical theory, also called geometric optics, describing the rectilinear propagation, reflection, and refraction of light rays. A brief discussion of the latter ideas is presented in the introductory chapter that groups together elementary examples that could be, but are usually not, part of a partial differential equations course. The WKB expansions of geometric optics also play a crucial role in understanding the connection of classical and quantum mechanics. That example, though not hyperbolic, is presented in §5.2.2.

The theory of hyperbolic mixed initial boundary value problems, a subject with many interesting applications and many difficult challenges, is not discussed. Nor is the geometric optics approach to shocks.

I have omitted several areas where there are already good sources; for example, the books [Smoller, 1983], [Serre, 1999], [Serre, 2000], [Dafermos, 2010], [Majda, 1984], [Bressan, 2000] on conservation laws, and the books [Hörmander, 1997] and [Taylor, 1991] on the use of pseudodifferential techniques in nonlinear problems. Other books on hyperbolic partial differential equations include [Hadamard, 1953], [Leray, 1953], [Mizohata, 1965], and [Benzoni-Gavage and Serre, 2007]. Lax's 1963 Stanford notes occupy a special place for me. I took a course from him in the late 1960s that corresponded to the enlarged version [Lax, 2006]. When I approached him to ask if he'd be my thesis director he asked what interested me. I indicated two subjects from the course, mixed initial boundary value problems and the section on waves and rays. The first became the topic of my thesis, and the second is the subject of this book and at the core of much of my research. I owe a great intellectual debt to the lecture notes, and to all that Peter Lax has taught me through the years.

The book introduces a large and rich subject and I hope that readers are sufficiently attracted to probe further.

P.2. A bird's eye view of hyperbolic equations

The central theme of this book is hyperbolic partial differential equations. These equations are important for a variety of reasons that we sketch here and that recur in many different guises throughout the book.

The first encounter with hyperbolicity is usually in considering scalar real linear second order partial differential operators in two variables with coefficients that may depend on x,

$$a\, u_{x_1 x_1} \ + \ b\, u_{x_1 x_2} \ + \ c\, u_{x_2 x_2} \ + \ \text{lower order terms}\,.$$

Associate the quadratic form $\xi \mapsto a\,\xi_1^2 + b\,\xi_1 \xi_2 + c\,\xi_2^2$. The differential operator is *elliptic* when the form is positive or negative definite. The differential operator is *strictly hyperbolic* when the form is indefinite and nondegenerate.[1]

In the elliptic case one has strong local regularity theorems and solvability of the Dirichlet problem on small discs. In the hyperbolic cases, the initial value problem is locally well set with data given at noncharacteristic curves and there is finite speed of propagation. Singularities or oscillations in Cauchy data propagate along characteristic curves.

The defining properties of hyperbolic problems include well posed Cauchy problems, finite speed of propagation, and the existence of wave like structures with infinitely varied form. To see the latter, consider in $\mathbb{R}^2_{t,x}$ initial data on $t = 0$ with the form of a short wavelength wave packet, $a(x)\, e^{ix/\epsilon}$, localized near a point p. The solution will launch wave packets along each of two characteristic curves. The envelopes are computed from those of the initial data, as in §5.2, and can take any form. One can send essentially arbitrary amplitude modulated signals.

The infinite variety of wave forms makes hyperbolic equations the preferred mode for communicating information, for example in hearing, sight, television, and radio. The model equations for the first are the linearized compressible inviscid fluid dynamics, a.k.a. acoustics. For the latter three it is Maxwell's equations. The telecommunication examples have the property that there is propagation with very small losses over large distances. The examples of wave packets and long distances show the importance of short wavelength and large time asymptotic analyses.

[1] The form is nondegenerate when its defining symmetric matrix is invertible.

Well posed Cauchy problems with finite speed lead to hyperbolic equations.[2] Since the fundamental laws of physics must respect the principles of relativity, finite speed is required. This together with causality requires hyperbolicity. Thus there are many equations from physics. Those which are most fundamental tend to have close relationships with Lorentzian geometry. D'Alembert's wave equation and the Maxwell equations are two examples. Problems with origins in general relativity are of increasing interest in the mathematical community, and it is the hope of hyperbolicians that the wealth of geometric applications of elliptic equations in Riemannian geometry will one day be paralleled by Lorentzian cousins of hyperbolic type.

A source of countless mathematical and technological problems of hyperbolic type are the equations of inviscid compressible fluid dynamics. Linearization of those equations yields linear acoustics. It is common that viscous forces are important only near boundaries, and therefore for many phenomena inviscid theories suffice. Inviscid models are often easier to compute numerically. This is easily understood as a small viscous term $\epsilon^2\partial^2/\partial x^2$ introduces a length scale $\sim\epsilon$, and accurate numerics require a discretization small enough to resolve this scale, say $\sim\epsilon/10$. In dimensions $1+d$ discretization of a unit volume for times of order 1 on such a scale requires $10^4\epsilon^{-4}$ mesh points. For ϵ only modestly small, this drives computations beyond the practical. Faced with this, one can employ meshes which are only locally fine or try to construct numerical schemes which resolve features on longer scales without resolving the short scale structures. Alternatively, one can use asymptotic methods like those in this book to describe the boundary layers where the viscosity cannot be neglected (see for example [Grenier and Guès, 1998] or [Gérard-Varet, 2003]). All of these are active areas of research.

One of the key features of inviscid fluid dynamics is that smooth large solutions often break down in finite time. The continuation of such solutions as nonsmooth solutions containing shock waves satisfying suitable conditions (often called entropy conditions) is an important subarea of hyperbolic theory which is not treated in this book. The interested reader is referred to the conservation law references cited earlier. An interesting counterpoint is that for suitably dispersive equations in high dimensions, small smooth data yield global smooth (hence shock free) solutions (see §6.7).

The subject of geometric optics is a major theme of this book. The subject begins with the earliest understanding of the propagation of light. Observation of sunbeams streaming through a partial break in clouds or a

[2]See [Lax, 2006] for a proof in the constant coefficient linear case. The necessity of hyperbolicity in the variable coefficient case dates to [Lax, Duke J., 1957] for real analytic coefficients. The smooth coefficient case is due to Mizohata and is discussed in his book.

flashlight beam in a dusty room gives the impression that light travels in straight lines. At mirrors the lines reflect with the usual law of equal angles of incidence and reflection. Passing from air to water the lines are bent. These phenomena are described by the three fundamental principles of a physical theory called *geometric optics*. They are rectilinear propagation and the laws of reflection and refraction.

All three phenomena are explained by Fermat's principle of least time. The rays are locally paths of least time. Refraction at an interface is explained by positing that light travels at different speeds in the two media. This description is purely geometrical involving only broken rays and times of transit. The appearance of a minimum principle had important philosophical impact, since it was consistent with a world view holding that nature acts in a best possible way. Fermat's principle was enunciated twenty years before Römer demonstrated the finiteness of the speed of light based on observations of the moons of Jupiter.

Today light is understood as an electromagnetic phenomenon. It is described by the time evolution of electromagnetic fields, which are solutions of a system of partial differential equations. When quantum effects are important, this theory must be quantized. A mathematically solid foundation for the quantization of the electromagnetic field in $1 + 3$ dimensional space time has not yet been found.

The reason that a field theory involving partial differential equations can be replaced by a geometric theory involving rays is that visible light has very short wavelength compared to the size of human sensory organs and common physical objects. Thus, much observational data involving light occurs in an asymptotic regime of very short wavelength. The short wavelength asymptotic study of systems of partial differential equations often involves significant simplifications. In particular there are often good descriptions involving rays. We will use the phrase *geometric optics* to be synonymous with short wavelength asymptotic analysis of solutions of systems of partial differential equations.

In optical phenomena, not only is the wavelength short but the wave trains are long. The study of structures which have short wavelength and are in addition very short, say a short pulse, also yields a geometric theory. Long wave trains have a longer time to allow nonlinear interactions which makes nonlinear effects more important. Long propagation distances also increase the importance of nonlinear effects. An extreme example is the propagation of light across the ocean in optical fibers. The nonlinear effects are very weak, but over 5000 kilometers, the cumulative effects can be large. To control signal degradation in such fibers, the signal is treated about every 30 kilometers. Still, there is free propagation for 30 kilometers which needs to be understood. This poses serious analytic, computational, and engineering challenges.

A second way to bring nonlinear effects to the fore is to increase the amplitude of disturbances. It was only with the advent of the laser that sufficiently intense optical fields were produced so that nonlinear effects are routinely observed. The conclusion is that for nonlinearity to be important, either the fields or the propagation distances must be large. For the latter, dissipative losses must be small.

The ray description as a simplification of the Maxwell equations is analogous to the fact that classical mechanics gives a good approximation to solutions of the Schrödinger equation of quantum mechanics. The associated ideas are called the *quasiclassical approximation*. The methods developed for hyperbolic equations also work for this important problem in quantum mechanics. A brief treatment is presented in §5.2.2. The role of rays in optics is played by the paths of classical mechanics. There is an important difference in the two cases. The Schrödinger equation has a small parameter, Planck's constant. The quasiclassical approximation is an approximation valid for small Planck's constant. The mathematical theory involves the limit as this constant tends to zero. The Maxwell equations apparently have a small parameter too, the inverse of the speed of light. One might guess that rays occur in a theory where this speed tends to infinity. This is not the case. For the Maxwell equations in a vacuum the small parameter that appears is the wavelength which is introduced via the initial data. It is not in the equation. The equations describing the dispersion of light when it interacts with matter do have a small parameter, the inverse of the resonant frequencies of the material, and the analysis involves data tuned to this frequency just as the quasiclassical limit involves data tuned to Planck's constant. Dispersion is one of my favorite topics. Interested readers are referred to the articles [Donnat and Rauch, 1997] (both) and [Rauch, 2007].

Short wavelength phenomena cannot simply be studied by numerical simulations. If one were to discretize a cubic meter of space with mesh size 10^{-5} cm so as to have five mesh points per wavelength, there would be 10^{21} data points in each time slice. Since this is nearly as large as the number of atoms per cubic centimeter, there is no chance for the memory of a computer to be sufficient to store enough data, let alone make calculations. Such brute force approaches are doomed to fail. A more intelligent approach would be to use radical local mesh refinement so that the fine mesh was used only when needed. Still, this falls far outside the bounds of present computing power. Asymptotic analysis offers an alternative approach that is not only powerful but is mathematically elegant. In the scientific literature it is also embraced because the resulting equations sometimes have exact solutions and scientists are well versed in understanding phenomena from small families of exact solutions.

Short wavelength asymptotics can be used to great advantage in many disparate domains. They explain and extend the basic rules of linear geometric optics. They explain the dispersion and diffraction of linear electromagnetic waves. There are nonlinear optical effects, generation of harmonics, rotation of the axis of elliptical polarization, and self-focusing, which are also well described.

Geometric optics has many applications within the subject of partial differential equations. They play a key role in the problem of solvability of linear equations via results on propagation of singularities as presented in §5.5. They are used in deriving necessary conditions, for example for hypoellipticity and hyperbolicity. They are used by Ralston to prove necessity in the conjecture of Lax and Phillips on local decay. Via propagation of singularities they also play a central role in the proof of sufficiency. Propagation of singularities plays a central role in problems of observability and controlability (see §5.6). The microlocal elliptic regularity theorem and the propagation of singularities for symmetric hyperbolic operators of constant multiplicity is treated in this book. These are the two basic results of linear microlocal analysis. These notes are not a systematic introduction to that subject, but they present an important part *en passant*.

Chapters 9 and 10 are devoted to the phenomenon of *resonance* whereby waves with distinct phases interact nonlinearly. They are preparatory for Chapter 11. That chapter constructs a family of solutions of the compressible $2d$ Euler equations exhibiting three incoming wave packets interacting to generate an infinite number of oscillatory wave packets whose velocities are dense in the unit circle.

Because of the central role played by rays and characteristic hypersurfaces, the analysis of conormal waves is closely related to geometric optics. The reader is referred to the treatment of progressing waves in [Lax, 2006] and to [Beals, 1989] for this material.

Acknowledgments. I have been studying hyperbolic partial differential equations for more that forty years. During that period, I have had the pleasure and privilege to work for extended periods with (in order of appearance) M. Taylor, M. Reed, C. Bardos, G. Métivier, G. Lebeau, J.-L. Joly, and L. Halpern. I thank them all for the things that they have taught me and the good times spent together. My work in geometric optics is mostly joint with J.-L. Joly and G. Métivier. This collaboration is the motivation and central theme of the book. I gratefully acknowledge my indebtedness to them.

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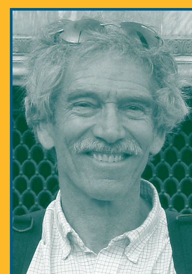
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This book introduces graduate students and researchers in mathematics and the sciences to the multifaceted subject of the equations of hyperbolic type, which are used, in particular, to describe propagation of waves at finite speed.

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One of the strengths of this book is its careful motivation of ideas and proofs, showing how they evolve from related, simpler cases. This makes the book quite useful to both researchers and graduate students interested in hyperbolic partial differential equations. Numerous exercises encourage active participation of the reader.

The author is a professor of mathematics at the University of Michigan. A recognized expert in partial differential equations, he has made important contributions to the transformation of three areas of hyperbolic partial differential equations: nonlinear microlocal analysis, the control of waves, and nonlinear geometric optics.

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